Range characterization of Radon transforms and ultrahyperbolic type of differential operators(Dissertation_全文)

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Range characterization of Radon transforms and ultrahyperbolic type of differential operators

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Introduction.

The characterization of the ranges of Radon transforms is one of the most important subjects in integral geometry. In fact, for the Radon transforms on Euclidean spaces, this subject has been studied from the several points of view since John's result [9], in which John showed that the range of the X-ray Radon transform on the 3-dimensional Euclidean space $\mathbf{R}^3$ is characterized as a kernel of some second order ultrahyperbolic differential operator.

On the other hand, the range-characterization of Radon transforms on compact symmetric spaces was first treated by Grinberg [5], in which he characterized the range of the Radon transform on a real or complex projective space by some invariant system of second order differential operators, using a representation theoretical method.

Our interest however lies in the explicit form of the range-characterizing operator, which Grinberg did not study in detail. We shall show in this paper that, as in the Euclidean case, the range of the Radon transform on a projective space can be also represented as a kernel of a certain ultrahyperbolic type of invariant differential operator on a corresponding Grassmann manifold.

To specify the meaning of the *ultrahyperbolic* type of differential operator and to explain the outline of this paper, we begin with the Funk transform on the standard 3-dimensional sphere $\mathbf{S}^3$ with radius 1, which is a special case treated in our previous paper, Kakehi and Tsukamoto [11]. Here the Funk transform on $\mathbf{S}^3$ is a Radon transform defined by averaging functions on $\mathbf{S}^3$ over oriented great circles. We denote the above Radon transform by R and the set of all the oriented great circles by M. Then R can be written as follows.

$$(Rf)(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} f(\gamma(s))ds, \quad \text{for } f \in C^\infty(\mathbf{S}^3), \text{ and } \gamma \in M,$$

where $\gamma(s)$ is the parametrization of γ by its arclength. We find easily that R is a mapping from $C^\infty(\mathbf{S}^3)$ to $C^\infty(M)$.

Since the dimension of M is 4 and is greater than the dimension of $\mathbf{S}^3$, we cannot expect the Radon transform R to be surjective. Thus, we try to find a good characterization of the range of R.

The special orthogonal group $SO(4)$ acts on M transitively. We put $G = SO(4)$ and denote by K the isotropy subgroup of G at the great circle $\gamma_0 = \{(\cos s, \sin s, 0, 0); 0 \le s \le 2\pi\} \in M$. We find that $K \cong SO(2) \times SO(2)$. Thus, we can consider M as a symmetric space G/K with the standard G-invariant metric. Moreover, we can identify the space $C^\infty(M)$ with a subspace of $C^\infty(G)$ consisting of the functions f which satisfy $f(gk) = f(g)$ for all $g \in G$ and $k \in K$.

We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively, and take $\{X_{ij}\}_{i=3,4\, j=1,2}$ as a basis of the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$, where X_{ij} is a 4×4 matrix whose (k,l) entry is given by $\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

An element $X \in \mathfrak{g}$ is considered to be a left invariant vector field acting on the space $C^\infty(G)$ as follows.

$$(Xf)(g) = \frac{d}{dt} f(g \exp tX)|_{t=0} \quad \text{for } f \in C^\infty(G), \text{ and } g \in G.$$

Using the above notation, we define a second order differential operator P acting on $C^\infty(G)$ by

$$P = X_{31}X_{42} - X_{32}X_{41}.$$

It is easily checked that P is invariant under the adjoint representation of K, that is, $(Pf)(gk) = (Pf)(g)$ for $f \in C^\infty(M)$ and for $g \in G$, $k \in K$. Thus, P can be regarded as an invariant differential operator acting on $C^\infty(M)$.

After an easy computation, we have for $f \in C^\infty(\mathrm{S}^3)$,

$$\begin{aligned}(X_{31}X_{42}Rf)(\gamma_0) &= (X_{32}X_{41}Rf)(\gamma_0)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos s \sin s\, (\nabla_{\mathbf{e}_3}\nabla_{\mathbf{e}_4} f)(\cos s, \sin s, 0, 0)\, ds,\end{aligned}$$

where $\mathbf{e}_i$ denotes the unit vector $\in \mathbf{R}^4$ whose k-th component is given by δ_{ik} and $\nabla_{\mathbf{e}_i}$ denotes the covariant derivative with respect to $\mathbf{e}_i$.

Thus, we have $(PRf)(\gamma_0) = 0$.

In the same way, we have $(PRf)(\gamma) = 0$ for any other great circle $\gamma \in M$. Therefore, we obtain $PRf = 0$ for all $f \in C^\infty(\mathrm{S}^3)$, which means that the range of R is included in the kernel of P

In fact, the range of R is identical with the kernel of P Therefore, in this case, the range is characterized by the second order differential operator P

Moreover, the differential operator P is ultrahyperbolic and of the similar form to John's ultrahyperbolic operator. Indeed, since we notice that $[X_{31}, X_{42}] = [X_{32}, X_{41}] = 0$, we find that P can be rewritten as follows.

$$Pf(g) = \left(\frac{\partial^2}{\partial x_{31}\partial x_{42}} - \frac{\partial^2}{\partial x_{32}\partial x_{41}}\right) f(g \exp(\sum_{i=3,4\, j=1,2} x_{ij}X_{ij}))|_{x_{ij}=0}.$$

The above representation of P means that P is ultrahyperbolic.

Taking this into account, we can say that the characterization by P corresponds to John's characterization.

When we extend the above result to the case of the Funk transform on the n-dimensional sphere $\mathbf{S}^n$ for $n \geq 4$, we need to represent the range-characterizing operator as a vector-bundle valued second order differential operator or a single fourth order differential operator. (See [11].)

In this paper, we deal with the Radon transforms on three kinds of projective spaces, the n-dimensional real projective space $\mathbf{P}^n\mathbf{R}$, the n-dimensional complex projective space $\mathbf{P}^n\mathbf{C}$, and the n-dimensional quaternionic projective space $\mathbf{P}^n\mathbf{H}$.

For example, we define a projective l-plane Radon transform on $\mathbf{P}^n\mathbf{R}$ by averaging functions on $\mathbf{P}^n\mathbf{R}$ over projective l-planes.

For other two kinds of projective spaces, a projective l-plane Radon transform can be defined in the same way as above.

We find that the projective l-plane Radon transform is a mapping from the space of smooth functions on each projective space to the space of smooth functions on a corresponding Grassmann manifold. It is well-known that the projective $(n-1)$-plane Radon transform is an isomorphism. (See Helgason [7].) Moreover, in the case $1 \leq l \leq n-2$, it is known that the projective l-plane Radon transform is injective. However, in this case, we can no longer expect its surjectivity by the same reason as in the case of the Funk transform. Therefore, we arrive at the problem of the range-characterization for the projective l-plane Radon transform.

Our main result is, roughly speaking, given by the following.

Theorem. *For $1 \leq l \leq n-2$ and for the projective l-plane Radon transform R on each n-dimensional projective space, there exists an invariant differential operator P of ultrahyperbolic type on a corresponding Grassmann manifold such that the range of R is identical with the kernel of P, that is,* $\operatorname{Ker} P = \operatorname{Im} R$.

Remark. As is well-known, there are five kinds of compact rank one symmetric spaces; $\mathbf{S}^n$, $\mathbf{P}^n\mathbf{R}$, $\mathbf{P}^n\mathbf{C}$, $\mathbf{P}^n\mathbf{H}$, and the Cayley projective plane $\mathbf{P}^2\mathrm{Cay}$. It is well-known that the Radon transform on Cayley projective plane, which is defined by averaging functions on $\mathbf{P}^2\mathrm{Cay}$ over Cayley projective lines, maps isomorphically the smooth functions on $\mathbf{P}^2\mathrm{Cay}$ to the smooth functions on the manifold of all the antipodal manifolds. Thus, we may not consider the range-characterization of the Radon transform in this case. Furthermore, it is easily understood that the consideration of the Radon transform on $\mathbf{S}^n$ is equivalent to that on $\mathbf{P}^n\mathbf{R}$. Therefore, the above theorem and Helgason's results give the answer to the problem of range characterization for Radon transforms on compact rank one symmetric spaces.

This paper is organized as follows.

In Chapter 1, we will prove the main theorem in the case of $\mathbf{P}^n\mathbf{C}$. Furthermore, we will represent the range characterizing operator as an ultrahyperbolic type of differential operator.

In Chapter 2, we will deal with two kinds of Radon transforms. One is the projective l-plane Radon transform on $\mathbf{P}^n\mathbf{R}$ and the other is a Radon transform on $\mathbf{S}^n$ defined by averaging functions on $\mathbf{S}^n$ over oriented totally geodesic l-dimensional spheres. We will represent the range-characterizing operator in the similar form to that in Chapter 1.

In Chapter 3, we will deal with the case of $\mathbf{P}^n\mathbf{H}$. However, the author does not represent the range-characterizing operator in the similar form to that in Chapter 1 or in Chapter 2. In this case, the range characterizing operator will be constructed in a different way.

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Chapter 1.

Range characterization of Radon transforms on complex projective spaces

We deal with the case of $\mathbf{P}^n\mathbf{C}$ in this chapter. The purpose of this chapter is to characterize the ranges of Radon transforms on $\mathbf{P}^n\mathbf{C}$ by invariant differential operators of ultrahyperbolic type.

§1. Range-characterizing operator.

Let M be the set of all $(l+1)$-dimensional complex vector subspaces of $\mathbf{C}^{n+1}$, that is, the set of projective l-planes in $\mathbf{P}^n\mathbf{C}$. Then M is a compact symmetric space $SU(n+1)/S(U(l+1)\times U(n-l))$ of rank $\min\{l+1, n-l\}$. We assume that $r := \operatorname{rank} M \geq 2$, that is, $1 \leq l \leq n-2$.

We define a projective l-plane Radon transform $R : C^\infty(\mathbf{P}^n\mathbf{C}) \to C^\infty(M)$ by

$$Rf(\xi) = \frac{1}{\operatorname{Vol}(\mathbf{P}^l\mathbf{C})} \int_{x\in\xi} f(x)dv_\xi(x), \quad \xi \in M, \quad f \in C^\infty(\mathbf{P}^n\mathbf{C}),$$

where $dv_\xi(x)$ denotes the measure on $\xi\,(\subset \mathbf{P}^n\mathbf{C})$ induced by the canonical measure on $\mathbf{P}^n\mathbf{C}$.

We proceed to write down a range-characterizing operater.

For a Lie group G and its closed subgroup H, we denote by $C^\infty(G,H)$ the set $\{\, f \in C^\infty(G); f(gh) = f(g) \quad \forall g \in G \text{ and } \forall h \in H \,\}$, and we identify $C^\infty(G,H)$ with $C^\infty(G/H)$. We define an action L_g of G on $C^\infty(G)$ by $(L_g f)(x) = f(g^{-1}x)$ for $x \in G$, and $f \in C^\infty(G)$. Similarly we define an action R_g of G on $C^\infty(G)$ by $(R_g f)(x) = f(xg)$. A differential operator D on G is called left-G-invariant if $L_g D = D L_g$ for all $g \in G$. Similarly, D is called right-H-invariant if $R_h D = D R_h$ for all $h \in H$. We identify a right-H-invariant differential operator on G with a differential operator on G/H.

Let G, K, and K' be the groups $SU(n+1)$, $S(U(l+1)\times U(n-l))$, and $S(U(1)\times U(n))$, respectively. Then we have $M = G/K$, $\mathbf{P}^n\mathbf{C} = G/K'$, and, by the above identification, $C^\infty(G,K) = C^\infty(M)$, $C^\infty(G,K') = C^\infty(\mathbf{P}^n\mathbf{C})$. We choose a Killing form metric on G, which induces metrics on K, K', M, and $\mathbf{P}^n\mathbf{C}$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively.

$$\mathfrak{g} = \{\, X \in M_{n+1}(\mathbf{C}); X + X^* = 0,\ \operatorname{tr} X = 0 \,\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g}; X_1 \in M_{l+1}(\mathbf{C}),\ X_2 \in M_{n-l}(\mathbf{C}) \right\}$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition, where $\mathfrak{m}$ is all the matrices of the form

$$Z = \begin{pmatrix} 0 & \cdots & 0 & -\bar{z}_{l+2,1} & & -\bar{z}_{n+1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\bar{z}_{l+2,l+1} & \cdots & -\bar{z}_{n+1,l+1} \\ z_{l+2,1} & \cdots & z_{l+2,l+1} & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n+1,1} & \cdots & z_{n+1,l+1} & 0 & \cdots & 0 \end{pmatrix}$$

We define second order differential operators $L_{ij,\alpha\beta}$ $(l+2 \le i < j \le n+1, 1 \le \alpha < \beta \le l+1)$ and a fourth order differential operator P on G as follows.

$$L_{ij,\alpha\beta} f(g) = \left(\frac{\partial^2}{\partial z_{i\alpha} \partial z_{j\beta}} - \frac{\partial^2}{\partial z_{i\beta} \partial z_{j\alpha}} \right) f(g \exp Z)|_{Z=0}, \quad f \in C^\infty(G),$$

$$P = \sum_{\substack{l+2 \le i < j \le n+1 \\ 1 \le \alpha < \beta \le l+1}} L^*_{ij,\alpha\beta} L_{ij,\alpha\beta},$$

where $L^*_{ij,\alpha\beta}$ denotes the adjoint operator of $L_{ij,\alpha\beta}$ and is given by

$$L^*_{ij,\alpha\beta} f(g) = \left(\frac{\partial^2}{\partial \bar{z}_{i\alpha} \partial \bar{z}_{j\beta}} - \frac{\partial^2}{\partial \bar{z}_{i\beta} \partial \bar{z}_{j\alpha}} \right) f(g \exp Z)|_{Z=0}.$$

Lemma 1.1. *P is a right-K-invariant differential operator.*

Proof. We define Ad-K-invariant polynomials $F_j(Z)$ $(j = 1, 2, \ldots)$ on $\mathfrak{m}$ by

$$\det(\lambda I + Z) = \lambda^{n+1} + F_1(Z)\lambda^{n-1} + F_2(Z)\lambda^{n-3} + \cdots$$

Then we have

$$F_2(Z) = \sum_{\substack{l+2 \le i < j \le n+1 \\ 1 \le \alpha < \beta \le l+1}} (\bar{z}_{i\alpha}\bar{z}_{j\beta} - \bar{z}_{i\beta}\bar{z}_{j\alpha})(z_{i\alpha}z_{j\beta} - z_{i\beta}z_{j\alpha}). \tag{1.1}$$

On the other hand, we have

$$R_k P R_{k^{-1}} = \sum_{\substack{l+2 \le i < j \le n+1 \\ 1 \le \alpha < \beta \le l+1}} R_k L^*_{ij,\alpha\beta} R_{k^{-1}} \circ R_k L_{ij,\alpha\beta} R_{k^{-1}},$$

where

$$R_k L_{ij,\alpha\beta} R_{k^{-1}} f(g) = \left(\frac{\partial^2}{\partial z_{i\alpha} \partial z_{j\beta}} - \frac{\partial^2}{\partial z_{i\beta} \partial z_{j\alpha}} \right) f(g \exp kZk^{-1})|_{Z=0},$$

$$R_k L^*_{ij,\alpha\beta} R_{k^{-1}} f(g) = \left(\frac{\partial^2}{\partial \bar{z}_{i\alpha} \partial \bar{z}_{j\beta}} - \frac{\partial^2}{\partial \bar{z}_{i\beta} \partial \bar{z}_{j\alpha}} \right) f(g \exp kZk^{-1})|_{Z=0},$$

for $f \in C^\infty(G)$ and $k \in K$.

Thus, we have only to prove that P is invariant under the linear transform $Z \mapsto kZk^{-1} = Ad_k Z$, which follows easily from the fact that the polynomial $F_2(Z)$ is Ad-K-invariant. ∎

It is obvious that P is left-G-invariant. Therefore, P is well-defined as an invariant differential operator on M

The purpose of this chapter is to prove the following theorem.

Theorem 1.2. *The range of the Radon transform R is characterized by the invariant differential operator P, that is,* $\operatorname{Ker} P = \operatorname{Im} R$.

Remark 1.3. The above differential operators $L_{ij,\alpha\beta}$ and $L^*_{ij,\alpha\beta}$ are of the form similar to the ultrahyperbolic operator $P = X_{31}X_{42} - X_{32}X_{41}$ in the introduction. In this sense, we can say that the range of the Radon transform on $\mathbf{P}^n\mathbf{C}$ is also characterized by an *ultrahyperbolic* differential operator.

§2. Proof that $\operatorname{Im} \mathbf{R} \subset \operatorname{Ker} \mathbf{P}$.

We first prove that $\operatorname{Im} R \subset \operatorname{Ker} P$ By the identification $C^\infty(G,K) = C^\infty(M)$ and $C^\infty(G,K') = C^\infty(\mathbf{P}^n\mathbf{C})$, we consider the Radon transform R as a map from $C^\infty(G,K')$ to $C^\infty(G,K)$. Then R is given by

$$(2.1)\qquad (Rf)(g) = \frac{1}{\operatorname{Vol}(K)} \int_{k \in K} f(gk)dk, \quad f \in C^\infty(G,K').$$

From this section, we use the representation of the form (2.1).

We define a bilinear form $\langle\ ,\ \rangle$ on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$ by $\langle u,v\rangle = \sum_{j=1}^{n+1} u_j v_j$ for $u = (u_1,\ldots,u_{n+1})$, $v = (v_1,\ldots,v_{n+1})$, and a function $h^m_{a,b} \in C^\infty(G)$ by $h^m_{a,b}(g) = \langle a, g\mathrm{e}_1\rangle^m \langle b, \overline{g\mathrm{e}_1}\rangle^m$, where $a,b \in \mathbf{C}^{n+1}$, $\mathrm{e}_1 = (1,0,\ldots,0) \in \mathbf{C}^{n+1}$ and m is a nonnegative integer. It is easily checked that $h^m_{a,b} \in C^\infty(G,K')$, that is, $h^m_{a,b} \in C^\infty(\mathbf{P}^n\mathbf{C})$. Moreover, the following lemma holds.

Lemma 2.1. *Let V_m denote the subspace of $C^\infty(\mathbf{P}^n\mathbf{C})$ generated by the set $\{\, h^m_{a,b}; \langle a,b\rangle = 0 \,\}$ Then V_m is the eigenspace of $\Delta_{\mathbf{P}^n\mathbf{C}}$, the Laplacian on $\mathbf{P}^n\mathbf{C}$, corresponding to the m-th eigenvalue and V_m is irreducible under the action of G.*

For the proof, see [10], §14.

Proposition 2.2. $\operatorname{Im} R \subset \operatorname{Ker} P$

Proof. Since P and R are G-invariant operators and $L_{g^{-1}} h^m_{a,b} = h^m_{g^*a, g^*b}$, we have

$$P(R(h^m_{a,b}))(g) = P(R(h^m_{g^*a,g^*b}))(I),$$

where I denotes the $(n+1) \times (n+1)$ identity matrix.

Since the direct sum $\bigoplus_{m=0}^{\infty} R(V_m)$ is dense in $\operatorname{Im} R$ in C^∞-topology, we have only to prove $P(R(h^m_{a,b}))(I) = 0$, or,

$$\begin{aligned}
&L_{ij,\alpha\beta}(R(h^m_{a,b}))(I) \\
&= \frac{1}{\mathrm{Vol}(K)} \left(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}} - \frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}} \right) \int_{k\in K} h^m_{a,b}((\exp Z)k)dk|_{Z=0} \\
&= 0.
\end{aligned}$$

Here we have

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}} - \frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}} \right) \{\langle a, (\exp Z)ke_1\rangle^m \langle b, \overline{(\exp Z)ke_1}\rangle^m\}|_{Z=0} \\
&= m(m-1)\langle a, ke_1\rangle^{m-2}\langle b, \overline{ke_1}\rangle^m \\
&\quad \times \left\{ \frac{\partial}{\partial z_{i\alpha}}\langle a, Zke_1\rangle \frac{\partial}{\partial z_{j\beta}}\langle a, Zke_1\rangle - \frac{\partial}{\partial z_{j\alpha}}\langle a, Zke_1\rangle \frac{\partial}{\partial z_{i\beta}}\langle a, Zke_1\rangle \right\} \\
&+ m\langle a, ke_1\rangle^{m-1}\langle b, \overline{ke_1}\rangle^{m-1} \\
&\quad \times \left(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}} - \frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}} \right) \{\frac{1}{2}\langle a, Z^2ke_1\rangle\langle b, \overline{ke_1}\rangle \\
&\qquad\qquad + \langle a, Zke_1\rangle\langle b, \overline{Zke_1}\rangle + \frac{1}{2}\langle a, ke_1\rangle\langle b, \overline{Z^2ke_1}\rangle\} \\
&= m(m-1)(a_i k_{\alpha 1} a_j k_{\beta 1} - a_i k_{\beta 1} a_j k_{\alpha 1})\langle a, ke_1\rangle^{m-2}\langle b, \overline{ke_1}\rangle^m = 0,
\end{aligned}$$

where $k \in K$, and k_{ij} denotes the (i,j) entry of k. (In the above computation, we used the fact that the polynomial $\langle b, \overline{Zke_1}\rangle$ on $\mathfrak{m}$ is a linear combination of $\bar{z}_{pq}$'s and the fact that the polynomials $\langle a, Z^2ke_1\rangle$, $\langle a, Zke_1\rangle\langle b, \overline{Zke_1}\rangle$, and $\langle b, \overline{Z^2ke_1}\rangle$ on $\mathfrak{m}$ consist only of the terms of the form (constant)$\times z_{pq}\bar{z}_{p'q'}$.)

Therefore, the assertion is verified. ∎

§3. The inversion formula.

We construct a continuous linear map $S : C^\infty(M) \to C^\infty(\mathbf{P}^n\mathbf{C})$ such that $SR = Id$, where Id denotes the identity map.

Let Ξ denote the set of $(n-1)$ dimensional complex projective subspaces of $\mathbf{P}^n\mathbf{C}$. Then we have $\Xi = SU(n+1)/S(U(n)\times U(1))$, and we put $K'' = S(U(n)\times U(1))$. We define a Radon transform $\mathcal{F} : C^\infty(\mathbf{P}^n\mathbf{C}) \to C^\infty(\Xi)$ and its dual Radon transform $\check{\mathcal{F}} : C^\infty(\Xi) \to C^\infty(\mathbf{P}^n\mathbf{C})$ by

$$\mathcal{F}f(g) = \frac{1}{\mathrm{Vol}(K'')} \int_{k''\in K''} f(gk'')dk'', \quad f \in C^\infty(G, K'),$$

$$\check{\mathcal{F}}\phi(g) = \frac{1}{\mathrm{Vol}(K')} \int_{k'\in K'} \phi(gk')dk', \quad \phi \in C^\infty(G, K'').$$

We define a polynomial $\Phi(t)$ of of degree $n-1$ by

$$\Phi(t) = c_n \prod_{j=1}^{n-1} \left(t + \frac{(n-j)j}{n+1} \right)$$

$$\text{where} \quad c_n = (n+1)^{n-1} \prod_{j=1}^{n-1} \{j(n-j)\}^{-1}$$

Theorem 3.1. (Helgason [7], Ch. 1, Theorem 4.11.) *We have the inversion formula*

$$\Phi(\Delta_{\mathbf{P}^n\mathbf{C}})\check{\mathcal{F}}\mathcal{F} = Id.$$

Proposition 3.2. *There exists an inversion map $S : C^\infty(M) \to C^\infty(\mathbf{P}^n\mathbf{C})$ such that $SR = Id$.*

Proof. We define a continuous linear map $\tilde{R} : C^\infty(M) \to C^\infty(\Xi)$ by

$$\tilde{R}f(g) = \frac{1}{\mathrm{Vol}(K'')} \int_{k'' \in K''} f(gk'')dk'', \quad f \in C^\infty(G, K).$$

Then it is easily checked that $\tilde{R}R = \mathcal{F}$ Therefore, if we put

$$S = c_n \Phi(\Delta_{\mathbf{P}^n\mathbf{C}})\check{\mathcal{F}}\tilde{R}, \tag{3.1}$$

we get $SR = Id$ by Theorem 3.1. ∎

§4. Representation of (G, K).

In this section, we describe the root, the weight, and the Weyl group of (G, K). Let $\mathfrak{a} \subset \mathfrak{m}$ be the set of all matrices of the form

$$H(t) = H(t_1, \dots, t_r) = \sqrt{-1} \begin{pmatrix} 0 & \cdots & 0 & t_1 & & & \\ \vdots & & \vdots & & & & \\ 0 & \cdots & 0 & & & t_r & \\ t_1 & & & 0 & \cdots & & 0 \\ & & & \vdots & & & \vdots \\ & & t_r & & & & \\ & & & 0 & \cdots & & 0 \end{pmatrix},$$

where we put $r = \operatorname{rank} M (= \operatorname{rank} G/K)$ in Section 1 and $t = (t_1, \dots, t_r) \in \mathbf{R}^r$ Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}$. We identify $\mathfrak{a}$ with $\mathbf{R}^r$ by the mapping $H(t) \mapsto t$.

Let $(\ ,\ )$ denote an invariant inner product on $\mathfrak{g}$ defined by

$$(X, Y) = -2(n+1)\operatorname{tr}(XY) \quad X, Y \in \mathfrak{g},$$

which is a minus signed Killing form on $\mathfrak{g}$.

For $\alpha \in \mathfrak{a}$, we set $\mathfrak{g}_\alpha := \{ X \in \mathfrak{g}^{\mathbf{C}}; [H, X] = \sqrt{-1}(\alpha, H)X \text{ for all } H \in \mathfrak{a} \}$, and α is called a root of $(\mathfrak{g}, \mathfrak{a})$ when $\mathfrak{g}_\alpha \neq \{0\}$. We put $m_\alpha = \dim_{\mathbf{C}} \mathfrak{g}_\alpha$, and call it the multiplicity of α.

We put $H_i = H(0, \ldots, \overset{(i)}{1}, \ldots, 0)$ $(1 \leq i \leq r)$. Then the roots of $(\mathfrak{g}, \mathfrak{a})$ and their multiplicities are given by the following table.

α	m_α	
$\pm\dfrac{1}{2(n+1)}H_j$	1	$(1 \leq j \leq r)$,
$\pm\dfrac{1}{4(n+1)}H_j$	$2(n+1-2r)$	$(1 \leq j \leq r)$,
$\pm\dfrac{1}{4(n+1)}(H_j \pm H_k)$	2	$(1 \leq j < k \leq r)$.

We fix a lexicographical order $<$ on $\mathfrak{a}$ such that $H_1 > \cdots > H_r > 0$. Then the positive roots are $\frac{1}{2(n+1)}H_j$, $\frac{1}{4(n+1)}H_j$, $(1 \leq j \leq r)$, $\frac{1}{4(n+1)}(H_j \pm H_k)$, $(1 \leq j < k \leq r)$. *The simple roots are* $\frac{1}{4(n+1)}(H_1 - H_2)$, $\frac{1}{4(n+1)}(H_2 - H_3)$, $\ldots$, $\frac{1}{4(n+1)}(H_{r-1} - H_r)$, $\frac{1}{4(n+1)}H_r$. We define the positive Weyl chamber $\mathcal{A}^+$ by $\{ t \in \mathbf{R}^r; 0 < t_j < \frac{\pi}{2}\ (1 \leq j \leq r), t_1 > \cdots > t_r \}$.

We set

$$\Omega((\exp H(t))K) := \Big| \prod_{\alpha\,:\,\text{positive root}} (e^{\sqrt{-1}(\alpha, H(t))} - e^{-\sqrt{-1}(\alpha, H(t))})^{m_\alpha} \Big|.$$

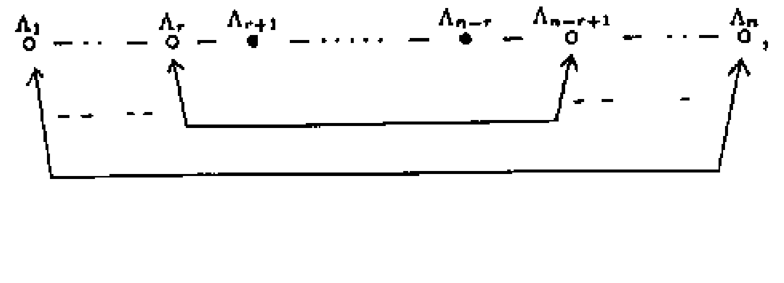
We consider Ω as a density function on $\mathbf{R}^r$, and we have

$$\begin{aligned} &\Omega = \sigma\omega^2, \\ \text{(4.1)} \qquad &\text{where} \quad \sigma = 2^{r(2n-2r+3)} \Big| \prod_{j=1}^{r} \sin 2t_j \sin^{2(n-r+1)} t_j \Big|, \\ &\omega = 2^{\frac{1}{2}r(r-1)} \prod_{j<k} (\cos 2t_j - \cos 2t_k). \end{aligned}$$

The Satake diagram of G/K is given by (4.2) or (4.3),

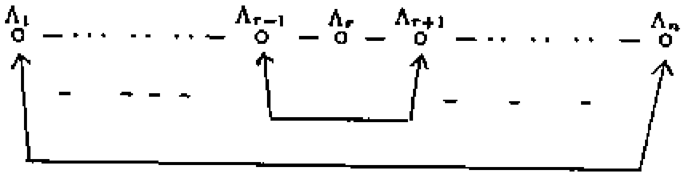
case A $\quad n+1 > 2r$:

(4.2)



case B $n+1=2r$:

(4.3)



In the diagram (4.2) or (4.3), $\Lambda_1, \ldots, \Lambda_n$ denote the fundamental weights of $\mathfrak{g}$, corresponding to the simple roots of $\mathfrak{g}$.

Since $\operatorname{rank} G/K = r$, there are r fundamental weights $M_1, \ldots, M_r$ of (G, K). By the diagram (4.2) or (4.3), $M_1, \ldots, M_r$ are given by

$$M_1 = \Lambda_1 + \Lambda_n, \ldots, M_{r-1} = \Lambda_{r-1} + \Lambda_{n-r+2}, M_r = \Lambda_r + \Lambda_{n-r+1}, \quad \text{(case A)},$$
$$M_1 = \Lambda_1 + \Lambda_n, \ldots, M_{r-1} = \Lambda_{r-1} + \Lambda_{n-r+2}, M_r = 2\Lambda_r, \quad \text{(case B)}.$$

Then we have
$$M_k = \frac{1}{2(n+1)} \sum_{j=1}^{k} H_j, \quad (1 \le k \le r).$$

Let $Z(G,K)$ be the weight lattice, that is, $Z(G,K) = \{ \frac{1}{4(n+1)}(\mu_1 H_1 + \cdots + \mu_r H_r); \mu_1, \ldots, \mu_r \in \mathbf{Z} \}$. The highest weight of a spherical representation of (G,K) is of the form $m_1 M_1 + \cdots + m_r M_r$, where $m_1, \ldots, m_r$ are non-negative integers. Let $V(m_1, \ldots, m_r)$ denote the eigenspace of the Laplacian Δ_M on G/K that is an irreducible representation space with the highest weight $m_1 M_1 + \cdots + m_r M_r$.

The Weyl group $W(G,K)$ of (G,K) is the set of all maps $s : \mathfrak{a} \to \mathfrak{a}$ such that

$$s : (t_1, \ldots, t_r) \mapsto (\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_r t_{\sigma(r)}), \quad \varepsilon_j = \pm 1, \ \sigma \in \mathfrak{S}_r.$$

§5. Radial part of P.

We calculate the radial part of the invariant differential operator P. The result in this section is used to calculate the eigenvalues of P in the next section.

To each invariant differential operator D on G/K, there corresponds a unique differential operator on Weyl chambers which is invariant under the action of the Weyl group $W(G,K)$. This operator is called a radial part of D, and we denote it by $\operatorname{rad}(D)$.

The following lemma is well-known. (See [12], Theorem 10.4.)

Lemma 5.1. *The radial part of the Laplacian Δ_M on M is given by*

$$\operatorname{rad}(\Delta_M) = -\frac{1}{4(n+1)} \sum_{j=1}^{r} \left(\frac{\partial^2}{\partial t_j{}^2} + \frac{\Omega_{t_j}}{\Omega} \frac{\partial}{\partial t_j} \right)$$

We define a differential operator Q_1 on $\mathbf{R}^r$ by

$$Q_1 := \frac{1}{\omega}\sum_{j=1}^{r}\left(\frac{\partial^2}{\partial t_j{}^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)\circ\omega.$$

The next lemma is easily checked.

Lemma 5.2.

$$-4(n+1)\,\mathrm{rad}(\Delta_M) = Q_1 - \sum_{j=1}^{r} 4j(j+n+2-2r).$$

We consider the following conditions (A), (B), (C), and (D) on a differential operator Q on $\mathbf{R}^r$ that is regular in all Weyl chambers.

(A) $\quad Q = \frac{1}{16}\sum_{j<k}\dfrac{\partial^2}{\partial t_j{}^2}\dfrac{\partial^2}{\partial t_k{}^2}$ + lower order terms.

(B) $\quad Q$ is formally self-adjoint with respect to the density Ωdt.

(C) $\quad Q$ is $W(G,K)$-invariant.

(D) $\quad [Q, \mathrm{rad}(\Delta_M)] := Q\,\mathrm{rad}(\Delta_M) - \mathrm{rad}(\Delta_M)Q = 0.$

Then the differential operator $\mathrm{rad}(P)$ satisfies the above four conditions (A), (B), (C), and (D). Indeed, the principal symbol of P is given by $\frac{1}{16}F_2(Z)$, which was defined in (1.1). (Notice that $\dfrac{\partial}{\partial z_{ij}} = \frac{1}{2}\dfrac{\partial}{\partial x_{ij}} - \frac{\sqrt{-1}}{2}\dfrac{\partial}{\partial y_{ij}}$ for $z_{ij} = x_{ij}+\sqrt{-1}y_{ij}$.) Therefore, its restriction to $\mathfrak{a}$ $\frac{1}{16}F_2(H(t))$ is $\frac{1}{16}\sum_{j<k} t_j^2 t_k^2$, and the condition (A) holds. The condition (B) follows from the self-adjointness of P Since P is an invariant differential operator, the conditions (C) and (D) are easily verified.

We define a differential operator Q_2 by

$$Q_2 := \frac{1}{16\omega}\sum_{j<k}\left(\frac{\partial^2}{\partial t_j{}^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)\left(\frac{\partial^2}{\partial t_k{}^2}+\frac{\sigma_{t_k}}{\sigma}\frac{\partial}{\partial t_k}\right)\circ\omega.$$

Lemma 5.3. *The differential operator Q_2 satisfies the conditions (A), (B), (C), and (D).*

Proof. The condition (A) is obvious. The condition (D) follows from Lemma 5.2. The conditions (B) and (C) are easily checked using the formula (4.3). ∎

Lemma 5.4. *If a differential operator Q satisfies the conditions (A), (B), (C), and (D), then Q can be written in the form*

$$Q = Q_2 + c_1\,\mathrm{rad}(\Delta_M) + c_2,$$

for suitable constants c_1, c_2.

Proof. Because of the conditions (A) and (B), $Q - Q_2$ is a second order differential operator and satisfies the conditions (B), (C), and (D). Therefore, the proof is reduced to the following lemma.

Lemma 5.5. *If a second order differential operator*

$$Q := \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial {t_j}^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j}$$

satisfies the conditions (B), (C), and (D), then Q can be written in the form

$$Q = c \operatorname{rad}(\Delta_M),$$

where c is a suitable constant.

Proof. By the condition (D), the third order terms of $[Q, \operatorname{rad}(\Delta_M)]$ vanish. Thus, we have the following equations.

$$A_{j,t_j} = 0, \qquad (1 \le j \le r); \tag{5.1}$$

$$A_{k,t_j} + B_{jk,t_k} = 0, \quad A_{j,t_k} + B_{jk,t_j} = 0, \qquad (j < k); \tag{5.2}$$

$$B_{ij,t_k} + B_{jk,t_i} + B_{ik,t_j} = 0, \qquad (1 \le i < j < k \le r). \tag{5.3}$$

By the equations (5.1), (5.2), and (5.3), we obtain

$$A_{j,t_k t_k t_k} = 0, \quad (j \neq k); \tag{5.4}$$

$$B_{jk,t_j t_j} = B_{jk,t_k t_k} = 0, \tag{5.5}$$

$$B_{jk,t_i t_i t_i} = 0, \quad (i \neq j, k). \tag{5.6}$$

By the condition (C) and the equations (5.1–6), the coefficients A_j and B_{jk} are polynomials of the form

$$A_j = \delta_1 \sum_{k \neq j} t_k^2 + \delta_2, \tag{5.7}$$

$$B_{jk} = -2\delta_1 t_j t_k, \tag{5.8}$$

where δ_1 and δ_2 are some constants.

Using the condition (B), we have

$$C_j = \frac{1}{\Omega}(A_j \Omega)_{t_j} + \frac{1}{2\Omega} \sum_{j<k} (B_{jk}\Omega)_{t_k} + \frac{1}{2\Omega} \sum_{k<j} (B_{kj}\Omega)_{t_k} \tag{5.9}$$

If $\delta_1 = 0$, then the coefficient $B_{jk} = 0$, and the coefficient $C_j = \delta_2 \Omega_{t_j}/\Omega$ by (5.9). Therefore, we obtain $Q = -4(n+1)\delta_2 \operatorname{rad}(\Delta_M)$, and the lemma holds.

Now, we suppose that $\delta_1 \neq 0$. Furthermore, we may suppose that $\delta_1 = 1$ and $\delta_2 = 0$. By the condition (D), the first order terms of $[Q, \mathrm{rad}(\Delta_M)]$ vanish. Thus, we have

$$Qa_j = -4(n+1)\,\mathrm{rad}(\Delta_M)C_j \quad (1 \leq j \leq r), \tag{5.10}$$

where we put $a_j = \Omega_{t_j}/\Omega$.

We extend the both sides of (5.10) to $\mathbf{C}$ as meromorphic functions of $t_1 = \mu_1 + \sqrt{-1}\nu_1$.

By the formula (4.1), we have

$$a_1 = 2\frac{\cos 2t_1}{\sin 2t_1} + 2(n-r+1)\frac{\cos t_1}{\sin t_1} + 2\sum_{j=2}^{r} \frac{-2\sin 2t_1}{\cos 2t_1 - \cos 2t_j}.$$

As $\nu_1 \to +\infty$, we have a_{1,t_j} and $a_{1,t_j t_k} \to 0$ (rapidly decreasing), and $a_1 = O(1)$. The same fact holds for a_j $(j = 2, \dots, r)$. Thus, $Qa_1 \to 0$ (rapidly decreasing), which means $\mathrm{rad}(\Delta_M)C_1 \to 0$ (rapidly decreasing) by (5.10). On the other hand, when ν_1 tends to $+\infty$, we have

$$\begin{aligned} -4(n+1)\,\mathrm{rad}(\Delta_M)C_1 &= \frac{1}{2}\sum_{j,k=2}^{r}\left(\frac{\partial^2}{\partial t_k{}^2} + a_k\frac{\partial}{\partial t_k}\right)(B_{1j}a_j + B_{1j,t_j}) + O(1) \\ &= -t_1\sum_{k=2}^{r} a_k{}^2 + O(1). \end{aligned}$$

(In the above computation, we used (5.7), (5.8) and the fact that $a_j = O(1)$ and the derivatives of $a_j \to 0$ as $\nu_1 \to \infty$.) Therefore, we have $\mathrm{rad}(\Delta_M)C_1 \to \infty$, for suitable $t_2, \dots, t_r$, and μ_1. It is a contradiction. ∎

Lemmas 5.1, 5.2, and 5.3 imply the following proposition.

Proposition 5.5. *The differential operator* $\mathrm{rad}(P)$ *can be expressed in the form*

$$\mathrm{rad}(P) = Q_2 + c_1 Q_1 + c_2,$$

for some constants c_1, c_2.

§6. Proof of Theorem 1.2.

We calculate the eigenvalue of P on $V(m_1, \dots, m_r)$ to prove Theorem 1.2.

Let $a(m_1, \dots, m_r)$ be the eigenvalue of P on $V(m_1, \dots, m_r)$ and $\phi_{(m_1,\dots,m_r)}$ the zonal spherical function which belongs to $V(m_1, \dots, m_r)$. We denote by $u_{(m_1,\dots,m_r)}$ the restriction of $\phi_{(m_1,\dots,m_r)}$ to the Weyl chamber A^+

Lemma 6.1. ([12], Theorem 8.1.) *The function $u_{(m_1,\dots,m_r)}$ has a Fourier series expansion on $\mathcal{A}^+$ of the form*

$$\begin{aligned}&u_{(m_1,\dots,m_r)}(t_1,\dots,t_r)\\&\quad=\sum_{\substack{\lambda\le m_1M_1+\cdots+m_rM_r\\ \lambda\in Z(G,K),\text{finite sum}}}\eta_\lambda\exp\sqrt{-1}(\lambda,t_1H_1+\cdots+t_rH_r),\end{aligned}$$

where $\eta_{m_1M_1+\cdots+m_rM_r}>0$.

Let f_1 and f_2 be Fourier series on $\mathcal{A}^+$ of the form

$$f_1=\sum_{\lambda\le\Lambda_1,\lambda\in Z(G,K)}\zeta_\lambda\exp\sqrt{-1}(\lambda,t_1H_1+\cdots+t_rH_r),$$

$$f_2=\sum_{\lambda\le\Lambda_2,\lambda\in Z(G,K)}\tilde{\zeta}_\lambda\exp\sqrt{-1}(\lambda,t_1H_1+\cdots+t_rH_r).$$

We denote $f_1\sim f_2$ when $\Lambda_1=\Lambda_2$ and $\zeta_{\Lambda_1}=\tilde{\zeta}_{\Lambda_2}(\neq 0)$. Obviously the relation $\sim$ is an equivalence relation.

Lemma 6.2. *We have the following relations.*

$$\sigma_{t_j}\sim 2\sqrt{-1}(n+2-2r)\sigma,\tag{6.1}$$

$$\omega_{t_j}\sim 2\sqrt{-1}(r-j)\omega,\tag{6.2}$$

$$\frac{\partial}{\partial t_j}u_{(m_1,\dots,m_r)}\sim 2\sqrt{-1}(m_j+m_{j+1}+\cdots+m_r)u_{(m_1,\dots,m_r)}.\tag{6.3}$$

Proof. The relations (6.1) and (6.2) are easily checked. The relation (6.3) follows from Lemma 6.1. ∎

Lemma 6.3.

$$\begin{aligned}&a(m_1,\dots,m_r)\\&\quad=\sum_{j<k}(l_j+r-j)(l_j+n+2-r-j)(l_k+r-k)(l_k+n+2-r-k)\\&\quad-4c_1\sum_{j=1}^{r}(l_j+r-j)(l_j+n+2-r-j)+c_2,\end{aligned}\tag{6.4}$$

where c_1 and c_2 are constants in Proposition 5.5, and $l_j=m_j+m_{j+1}+\cdots+m_r$.

Proof. Since $\phi_{(m_1,\dots,m_r)}\in V(m_1,\dots,m_r)$,

$$P\phi_{(m_1,\dots,m_r)}=a(m_1,\dots,m_r)\phi_{(m_1,\dots,m_r)}.\tag{6.5}$$

We restrict the both sides of (6.5) to $\mathcal{A}^+$, and then we have

$$\mathrm{rad}(P)u_{(m_1,\ldots,m_r)} = a(m_1,\ldots,m_r)u_{(m_1,\ldots,m_r)}.$$

By Proposition 5.4, we get

$$\begin{aligned}
&\sigma^2\omega \sum_{j<k} \frac{1}{\omega}\left(\frac{\partial^2}{\partial {t_j}^2} + \frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)\left(\frac{\partial^2}{\partial {t_j}^2} + \frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)(\omega u_{(m_1,\ldots,m_r)}) \\
&\quad + c_1\sigma^2\omega \sum_{j=1}^{r} \frac{1}{\omega}\left(\frac{\partial^2}{\partial {t_j}^2} + \frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)(\omega u_{(m_1,\ldots,m_r)}) \\
&\quad + c_2\sigma^2\omega u_{(m_1,\ldots,m_r)} \\
&\qquad = a(m_1,\ldots,m_r)\sigma^2\omega u_{(m_1,\ldots,m_r)}.
\end{aligned}$$

Using Lemma 6.2, we have

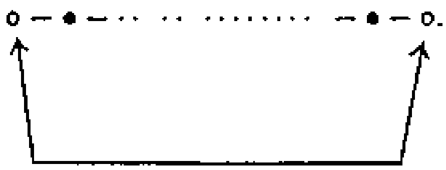
$$\begin{aligned}
&\sum_{j<k}\{((l_j+r-j)(l_j+n+2-r-j) \\
&\qquad \times (l_k+r-k)(l_k+n+2-r-k)\sigma^2\omega u_{(m_1,\ldots,m_r)}\} \\
&\quad - 4c_1\sum_{j=1}^{r}(l_j+r-j)(l_j+n+2-r-j)\sigma^2\omega u_{(m_1,\ldots,m_r)} \\
&\quad + c_2\sigma^2\omega u_{(m_1,\ldots,m_r)} \\
&\qquad \sim a(m_1,\ldots,m_r)\sigma^2\omega u_{(m_1,\ldots,m_r)}.
\end{aligned} \tag{6.6}$$

Comparing the leading coefficients of the both sides of (6.6), we get (6.4). ∎

Lemma 6.4. *$R : V_m \to V(m,0,\ldots,0)$ is an isomorphism.*

Proof. By Proposition 3.2, R is G-equivariant and one to one. Thus, we have only to prove that the highest weight of V_m is equal to that of $V(m,0,\ldots,0)$. The Satake diagram of $\mathbf{P}^n\mathbf{C}$ is given by

(6.7)



Comparing (6.7) with the diagram (4.2) or (4.3), we find that V_m corresponds to mM_1. On the other hand, the highest weight of $V(m,0,\ldots,0)$ is mM_1 by definition. This completes our proof. ∎

Now, we can calculate the eigenvalue of P by combining the above lemmas.

Theorem 6.5. *The eigenvalue $a(m_1,\dots,m_r)$ of P on $V(m_1,\dots,m_r)$ is given by*

$$\begin{aligned} &a(m_1,\dots,m_r)\\ &\quad=\sum_{j<k} l_j l_k(l_j+n+2-2j)(l_k+n+2-2k)\\ &\quad+\sum_{j=2}^{r}(j-1)(n+1-j)l_j(l_j+n+2-2j), \end{aligned} \tag{6.8}$$

where $l_j = m_j + m_{j+1} + \cdots + m_r$

Proof. By Proposition 2.2 and Lemma 6.4, we have $a(m,0,\dots,0) = 0$ for any non-negative integer m. Then by Lemma 6.3, we have

$$\begin{aligned} &(m+r-1)(m+n+1-r)\sum_{k=2}^{r}(r-k)(n+2-r-k)\\ &+\sum_{2\le j<k<r}(r-j)(n+2-r-j)(r-k)(n+2-r-k)\\ &-4c_1(m^2+nm)-4c_1\sum_{j=1}^{r}(r-j)(n+2-r-j)+c_2\\ &=0. \end{aligned}$$

Therefore, we get

$$c_1=\frac{1}{4}\sum_{k=3}^{r-1}(r-k)(n+2-r-k), \tag{6.9}$$

$$\begin{aligned} c_2&=\sum_{j<k}(r-j)(n+2-r-j)(r-k)(n+2-r-k)\\ &+\{\sum_{j=1}^{r-1}(r-j)(n+2-r-j)\}^2 \end{aligned} \tag{6.10}$$

Substituting (6.9) and (6.10) to (6.4), we obtain the formula (6.8). ∎

The following corollary is now obvious.

Corollary 6.6. *$V(m_1,\dots,m_r)$ is contained in* $\operatorname{Ker} P$, *if and only if $m_2 = \cdots = m_r = 0$.*

Proof of Theorem 1.2. Let $V := \bigoplus_{m=0}^{\infty} V(m,0,\dots,0)$ and $\tilde V := \bigoplus_{m=0}^{\infty} V_m$ (direct sums). Then we have $R:\tilde V \to V$ and $S: V\to \tilde V$. Here S is the inversion map defined in (3.1). Moreover, we have $SR = Id$ on $\tilde V$ and $RS = Id$ on V by Proposition 3.2 and Lemma 6.4.

By Corollary 6.6, V is dense in $\operatorname{Ker} P$ in C^∞-topology. Since the inversion map $S: C^\infty(M) \to C^\infty(\mathrm{P}^n\mathrm{C})$ is continuous, we have $RS = Id$ on $\operatorname{Ker} P$. This completes the proof. ∎

Remark 6.7 If $\phi \in \operatorname{Ker} P$, the inverse image of ϕ is given by $S\phi$, that is, $R(S\phi) = \phi$.

Remark 6.8. The invariant differential operator P, which we constructed in Section 1, is of least degree in all the invariant differential operators on M that characterize the range of R. It follows from the fact that the principal symbol $\frac{1}{16}F_2(Z)$ of P is of least degree in all the Ad-K-invariant polynomials on $\mathfrak{m}$ except for the principal symbol of the Laplacian.

Chapter 2.

Range characterization of Radon transforms on $\mathbf{S}^n$ and $\mathbf{P}^n\mathbf{R}$

In this chapter, we mainly deal with the Radon transform on the n-dimensional sphere $\mathbf{S}^n$ We first prove the range theorem for the Radon transform on $\mathbf{S}^n$, from which the range theorem for the Radon transform on $\mathbf{P}^n\mathbf{R}$ follows immediately.

§1. Range-characterizing operator.

We denote by $\widetilde{Gr}(l,n;\mathbf{R})$ the compact oriented real Grassmann manifold of all the oriented l-dimensional totally geodesic spheres in $\mathbf{S}^n$

In this chapter, we examine mainly the range of the Radon transform $R = R_l$ on the n-dimensional sphere $\mathbf{S}^n$ for $1 \le l \le n-2$, which we define by averaging a function f on $\mathbf{S}^n$ over an oriented l-dimensional totally geodesic sphere ξ, that is, we define R as follows.

$$Rf(\xi) = \frac{1}{\mathrm{Vol}(\mathbf{S}^l)} \int_{x\in\xi} f(x)dv_\xi(x),$$

where $dv_\xi(x)$ is the canonical measure on $\xi \subset \mathbf{S}^n$ This Radon transform R maps smooth functions on $\mathbf{S}^n$ to smooth functions on $\widetilde{Gr}(l,n;\mathbf{R})$, that is, $R : C^\infty(\mathbf{S}^n) \to C^\infty(\widetilde{Gr}(l,n;\mathbf{R}))$.

For simplicity, we put $M = \widetilde{Gr}(l,n;\mathbf{R})$ in this chapter. We find that M is a compact symmetric space of rank $\min\{l+1, n-l\}$. In this chapter, we assume that $r := \operatorname{rank} M \ge 2$, that is, $1 \le l \le n-2$ as in Chapter 1.

Let G, K, K' be the Lie groups $SO(n+1)$, $SO(l+1)\times SO(n-l)$, $SO(n)$, respectively in this chapter. Then we can consider $M = G/K$, $\mathbf{S}^n = G/K'$ in the usual manner. Thus, we can identify $C^\infty(G,K)$ with $C^\infty(M)$ and $C^\infty(G,K')$ with $C^\infty(\mathbf{S}^n)$, respectively. We define metrics on M, $\mathbf{S}^n$, G, K, and K' by the metrics induced from the Killing form metric on G, respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively.

$$\mathfrak{g} = \{X \in M_{n+1}(\mathbf{R}); X + {}^tX = 0,\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g}; X_1 \in M_{l+1}(\mathbf{R}), X_2 \in M_{n-l}(\mathbf{R}) \right\}$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition, where $\mathfrak{m}$ is the set of all the matrices of the form

$$X = \begin{pmatrix} 0 & \cdot\cdot & 0 & -x_{l+2,1} & \cdot\cdot & -x_{n+1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdot\cdot & 0 & -x_{l+2,l+1} & \cdot\cdot & -x_{n+1,l+1} \\ x_{l+2,1} & \cdot\cdot & x_{l+2,l+1} & 0 & \cdot\cdot & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+1,1} & \cdot\cdot & x_{n+1,l+1} & 0 & & 0 \end{pmatrix}$$

We define differential operators $L_{ij,\alpha\beta}$ $(l+2 \leq i < j \leq n+1, 1 \leq \alpha < \beta \leq l+1)$ on G by

$$L_{ij,\alpha\beta} = \left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta}\partial x_{j\alpha}}\right) f(g \exp X)|_{X=0}, \quad f \in C^\infty(G).$$

Using this, we define a differential operator P on G by

$$P = \begin{cases} L_{34,12} & \text{if } n = 3, l = 1, \\ \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} (L_{ij,\alpha\beta})^2 & \text{otherwise.} \end{cases}$$

Then P is right-K-invariant. Thus, P is well-defined as a differential operator on M. Its proof is the same as Lemma 1.1 in [10], and is reduced to the fact that the polynomial $F(X)$ on $\mathfrak{m}$ is Ad-K-invariant. Here

$$F(X) = \begin{cases} x_{31}x_{42} - x_{32}x_{41} & \text{if } n = 3, l = 1, \\ \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} (x_{i\alpha}x_{j\beta} - x_{i\beta}x_{j\alpha})^2 & \text{otherwise.} \end{cases}$$

We identify the principal symbol of P with $F(X)$.

By definition, P is left-G-invariant. Therefore, P is well-defined as an invariant differential operator on M. The main theorem of this chapter is the following.

Theorem 1.1. *The range of R is identical with the kernel of P, that is,*

$$\operatorname{Ker} P = \operatorname{Im} R.$$

Since we gave the proof for the case $l = 1$ in [11], we consider the other cases in this chapter.

§2. Proof that $\mathbf{Im\, R \subset Ker\, P}$

We first prove that $\operatorname{Im} R \subset \operatorname{Ker} P$ It is proved in the same way as the complex case (see [10]). By the identification $C^\infty(G, K) = C^\infty(M)$ and $C^\infty(G, K') =$

$C^\infty(\mathbf{S}^n)$, we consider R as a map from $C^\infty(G,K)$ to $C^\infty(G,K')$. Then R can be rewritten as

$$(2.1) \qquad (Rf)(g) = \frac{1}{\mathrm{Vol}(K)} \int_{k\in K} f(gk)dk, \quad f \in C^\infty(G,K').$$

From this section, we use the representation of the form (2.1).

We define a bilinear form $(\ ,\ )$ on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$ by $(u,v) = \sum_{j=1}^{n+1} u_j v_j$ for $u = (u_1, \ldots, u_{n+1})$, $v = (v_1, \ldots, v_{n+1})$, and a smooth function $h_a^m \in C^\infty(G)$ by $h_a^m(g) = (a, ge_1)^m$, where $a \in \mathbf{C}^{n+1}$, $e_1 = (1, 0, \ldots, 0)$ and m is a non-negative integer. It is easily checked that $h_a^m \in C^\infty(G,K')$, that is, $h_a^m \in C^\infty(\mathbf{S}^n)$. Moreover, the following lemma holds.

Lemma 2.1. *Let V_m denote the subspace of $C^\infty(\mathbf{S}^n)$ generated by the set $\{h_a^m;\ (a,a) = 0\}$. Then V_m is the eigenspace of $\Delta_{\mathbf{S}^n}$, the Laplacian of $\mathbf{S}^n$, corresponding to the m-th eigenvalue and V_m is irreducible under the action of G.*

For the proof, see [12].

We notice that we always consider the Laplacian on a compact manifold as a non-negative operator.

We will use the following proposition to calculate the eigenvalue of P in Section 6.

Proposition 2.2. $\mathrm{Im}\, R \subset \mathrm{Ker}\, P$

Proof. By Lemma 2.1 and by the same argument as in that of Proposition 2.2 in [10], we have only to prove that

$$\begin{aligned}
&L_{ij,\alpha\beta}(R(h_a^m))(I) \\
&= \frac{1}{\mathrm{Vol}(K)} \left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta}\partial x_{j\alpha}} \right) \int_{k\in K} h_a^m((\exp X)k)dk|_{X=0} \\
&= 0,
\end{aligned}$$

where I denotes an identity matrix.

The above result follows from the equation

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta}\partial x_{j\alpha}} \right) \{(a, (\exp X)ke_1)^m\}|_{X=0} \\
&= m(m-1)(a_i k_{\alpha 1} a_j k_{\beta 1} - a_i k_{\beta 1} a_j k_{\alpha 1})(a, ke_1)^{m-2} = 0,
\end{aligned}$$

where $k \in K$ and k_{ij} denotes the (i,j) entry of k. ∎

§3. The inversion formula.

We construct a continuous linear map $S : C^\infty(M) \to C^\infty(\mathbf{S}^n)$ such that $SR = Id$ on $C^\infty_{even}(\mathbf{S}^n)$, using the Helgason's inversion formula. Here Id denotes the identity map and $C^\infty_{even}(\mathbf{S}^n)$ denotes the space of all even functions in $C^\infty(\mathbf{S}^n)$. (The Radon transform R maps odd functions on $\mathbf{S}^n$ to 0.)

In this section, we denote by M_l the oriented real Grassmann manifold $SO(n+1)/SO(l+1)\times SO(n-l)$, by K_l the closed subgroup $SO(l+1)\times SO(n-l)$ of G, and by R_l the Radon transform $R : C^\infty(\mathbf{S}^n) \to C^\infty(M_l)$, respectively. We define a dual Radon transform $\widetilde{R_l} : C^\infty(M_l) \to C^\infty(\mathbf{S}^n)$ by

$$(\widetilde{R_l}f)(g) = \frac{1}{\mathrm{Vol}(K_l)}\int_{k\in K_l} f(gk)dk, \quad f \in C^\infty(G, K_l).$$

If k is even, we define a polynomial $\Phi_k(t)$ of degree $\frac{k}{2}$ by

$$\Phi_k(t) = c_{n,k}\prod_{j=1}^{\frac{k}{2}}\left(t + \frac{(k-2j+1)(n-k+2j+2)}{2n}\right),$$

$$\text{where} \quad c_{n,k} = (2n)^{\frac{k}{2}}\prod_{j=1}^{\frac{k}{2}}\{(k-2j+1)(n-k+2j+2)\}^{-1}$$

Then the Helgason's inversion formula for the Radon transform on $\mathbf{S}^n$ is given by the following

Theorem 3.1. (Helgason [7], Ch. 1, Theorem 4.5.) *If l is even, we have the inversion formula for R_l*

$$\Phi_l(\Delta_{\mathbf{S}^n})\widetilde{R_l}R_l = Id \quad on \quad C^\infty_{even}(\mathbf{S}^n),$$

Proposition 3.2. *There exists an inversion map $S = S_l : C^\infty(M_l) \to C^\infty(\mathbf{S}^n)$ such that $S_lR_l = Id$ on $C^\infty_{even}(\mathbf{S}^n)$*

Proof. If l is even, then Proposition 3.2 follows immediately from Theorem 3.1, and we may therefore prove this proposition in case l is odd. We define $R^l_{l+1} : C^\infty(M_l) \to C^\infty(M_{l+1}))$ by

$$(R^l_{l+1}f)(g) = \frac{1}{\mathrm{Vol}(K_{l+1})}\int_{k\in K_{l+1}} f(gk)dk, \quad f \in C^\infty(G, K_l)$$

Then it is easily checked that $R^l_{l+1}R_l = R_{l+1}$. Since l is odd, $l+1$ is even and there exists S_{l+1} such that $S_{l+1}R_{l+1} = Id$ on $C^\infty_{even}(\mathbf{S}^n)$. Therefore, if we put $S_l = S_{l+1}R^l_{l+1}$, we get $S_lR_l = Id$ on $C^\infty_{even}(\mathbf{S}^n)$. ∎

§4. Representation of (G, K).

In this section, we describe the root, the weight, and the Weyl group of the symmetric pair (G, K).

Let $\mathfrak{a} \subset \mathfrak{m}$ be the set of all matrices of the form

$$
H(t) = H(t_1, \ldots, t_r) = \begin{pmatrix}
0 & & 0 & -t_1 & & & & \\
\vdots & & \vdots & & & & & \\
0 & \cdots & 0 & & & -t_r & & \\
t_1 & & & 0 & & 0 & \cdots & 0 \\
& & & \vdots & & \vdots & & \vdots \\
& & t_r & 0 & & 0 & \cdots & 0 \\
& & & \vdots & & \vdots & & \vdots \\
& & & 0 & & 0 & \cdots & 0
\end{pmatrix},
$$

where $t = (t_1, \ldots, t_r) \in \mathbf{R}^r$ Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}$. We identify $\mathfrak{a}$ with $\mathbf{R}^r$ by the mapping $H(t) \mapsto t$.

Let $(\ ,\ )$ denote an invariant inner product on $\mathfrak{g}$ defined by

$$(X, Y) = -(n-1)\,\mathrm{tr}(XY) \qquad X, Y \in \mathfrak{g},$$

which is a minus-signed Killing form on $\mathfrak{g}$.

For $\alpha \in \mathfrak{a}$, let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^{\mathbf{C}}; [H, X] = \sqrt{-1}(\alpha, H)X \quad \text{for all } H \in \mathfrak{g}\}$$

An element $\alpha \in \mathfrak{g}$ is called a root of $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}_\alpha \neq \{0\}$. We put $m_\alpha = \dim_{\mathbf{C}} \mathfrak{g}_\alpha$ and call it the multiplicity of α.

We put $H_i = H(0, \ldots, \overset{(i)}{1}, \ldots, 0)$ $(1 \leq i \leq r)$ and we fix a lexicographical order $<$ on $\mathfrak{a}$ such that $H_1 > \cdots H_r > 0$. Then the positive root α of $(\mathfrak{g}, \mathfrak{a})$ and its multiplicity m_α are given by the following table.

α	m_α
$\frac{1}{2(n-1)}(H_j \pm H_k) \quad (1 \leq j < k \leq r)$	1
$\frac{1}{2(n-1)}H_j \quad (1 \leq j \leq r)$	$n+1-2r.$

The simple roots α_j $(1 \leq j \leq r)$ are given by the following table.

$$
\begin{aligned}
(n+1 > 2r): \qquad & \alpha_j = \frac{1}{2(n-1)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-1), \\
& \alpha_r = \frac{1}{2(n-1)}H_r. \\
(n+1 = 2r): \qquad & \alpha_j = \frac{1}{2(n-1)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-2), \\
& \alpha_{r-1} = \frac{1}{2(n-1)}(H_{r-1} + H_r) \\
& \alpha_r = \frac{1}{2(n-1)}(H_{r-1} - H_r).
\end{aligned}
$$

Let $M_j \quad (1 \le j \le r)$ be the fundamental weights of G/K corresponding to the simple roots α_j, $(1 \le j \le r)$. Then $M_j \quad (1 \le j \le r)$ are given by the following table.

$$(n+1 > 2r+1): \qquad M_j = \frac{1}{n-1}\sum_{k=1}^{j} H_k \quad (1 \le j \le r-1),$$

$$M_r = \frac{1}{2(n-1)}\sum_{k=1}^{r} H_k.$$

$$(n+1 = 2r \quad \text{or} \quad 2r+1): \qquad M_j = \frac{1}{n-1}\sum_{k=1}^{j} H_k.$$

If $n+1 > 2r$, the Weyl group $W(G,K)$ of (G,K) is the set of all maps $s : \mathfrak{a} \to \mathfrak{a}$ such that

$$(4.1) \qquad s : (t_1, \dots, t_r) \mapsto (\varepsilon_1 t_{\sigma(1)}, \dots, \varepsilon_r t_{\sigma(r)}) \quad \varepsilon_j = \pm 1, \quad \sigma \in \mathfrak{S}_r.$$

And if $n+1 = 2r$, $W(G,K)$ is the set of all maps s in (4.1) such that $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_r = 1$.

Let $Z(G,K)$ be the weight lattice generated by $\frac{1}{2(n-1)} H_j \ (1 \le j \le r)$. The highest weight of a spherical reprsentation of (G,K) is of the form $m_1 M_1 + \cdots + m_r M_r$, where $m_1, \dots, m_r$ are non-negative integers. We denote by $V(m_1, \dots, m_r)$ the eigenspace of Laplacian Δ_M on $M = G/K$ which is an irreducible representation space with the highest weight $m_1 M_1 + \cdots + m_r M_r$.

In the same manner, we can define a fundamental weight $M_1{}'$ of $(SO(n+1), SO(n))$, (that is, this is the case $l = 0$,) and we have

$$M_1{}' = \frac{1}{2(n-1)} H_1$$

Then $mM_1{}'$ is the highest weight of the m-th eigenspace V_m of the Laplacian Δ_{S^n}, which we defined in Section 2. It is easily checked that $2M_1{}'$ corresponds to M_1 by an adjoint action. Therefore, we get the following Lemma by Proposition 3.2.

Lemma 4.1. *The Radon transform R isomorphically maps the subspace V_{2m} of $C^\infty(\mathbf{S}^n)$ to the subspace $V(m, 0, \dots, 0)$ of $C^\infty(M)$.*

§5. Radial part of P.

We will calculate the eigenvalue of P on $V(m_1, \dots, m_r)$ to prove Theorem 1.1. There are two ways to calculate it. One is a representation theoretical approach, and the other is the method of radial part. We use the latter.

We define a density function Ω on $\mathfrak{a}$ by

$$\Omega(t) = \Big| \prod_{\alpha:\text{positive root}} 2\sin(\alpha, H(t))^{m_\alpha} \Big|.$$

Then $\Omega(t)$ is given by

$$\Omega(t) = c_{n,r}|\sigma\omega|, \tag{5.1}$$

where

$$\begin{aligned} c_{n,r} &= 2^{\frac{1}{2}r(2n+1-3r)}, \\ \sigma &= \prod_{j=1}^{r} \sin^{n+1-2r} t_j, \\ \omega &= \prod_{1\le j<k\le r} (\cos 2t_j - \cos 2t_k). \end{aligned} \tag{5.2}$$

We choose a connected component $\mathcal{A}^+$ of Weyl chambers such that $\sigma > 0$, $\omega > 0$ on $\mathcal{A}^+$ For example, we choose

$$\mathcal{A}^+ = \{(t_1, \dots, t_r) \in \mathbf{R}^r; 0 < t_1 < \cdots < t_r < \frac{\pi}{2}\} \quad (n+1 > 2r),$$
$$\mathcal{A}^+ = \{(t_1, \dots, t_r) \in \mathbf{R}^r; 0 < t_j \pm t_k < \pi, \quad 1 \le j < k \le r\} \quad (n+1 = 2r).$$

To each invariant differential operator D on G/K, there corresponds a unique differential operator on $\mathcal{A}^+$ which is invariant under the action of the Weyl group $W(G,K)$. This operator is called a radial part of D, and we denote it by $\mathrm{rad}(D)$.

The following lemma is well-known.

Lemma 5.1.

$$\mathrm{rad}(\Delta_M) = -\frac{1}{n-1}\sum_{j=1}^{r}\left(\frac{\partial^2}{\partial t_j{}^2} + \frac{\Omega_{t_j}}{\Omega}\frac{\partial}{\partial t_j}\right),$$

where Ω_{t_j} means a differentiation of Ω by t_j.

For the proof, see [12], Ch. 10, Cor. 1.

We calculate the radial part of P, for $l \ge 2$.

The differential operator $\mathrm{rad}(P)$ which is defined in Weyl chambers satisfies the following conditions.

(A) $\mathrm{rad}(P) = \sum_{1\le j<k\le r} \dfrac{\partial^4}{\partial t_j{}^2 \partial t_k{}^2} +$ lower order terms.

(B) $\mathrm{rad}(P)$ is formally self-adjoint with respect to the density Ωdt.

(C) $\mathrm{rad}(P)$ is $W(G,K)$-invariant.

(D) $[\mathrm{rad}(P), \mathrm{rad}(\Delta_M)] = 0$.

By the conditions (A) and (B), we get

$$\text{the third order terms of } \mathrm{rad}(P) = \sum_{j\ne k} \frac{\Omega_{t_k}}{\Omega}\frac{\partial^3}{\partial t_j{}^2 \partial t_k}$$

Thus, we can put

$$\begin{aligned}\operatorname{rad}(P) = &\sum_{j<k} \frac{\partial^4}{\partial t_j{}^2 \partial t_k{}^2} + \sum_{j \neq k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial t_j{}^2 \partial t_k} \\ &+ \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial t_j{}^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j},\end{aligned} \tag{5.3}$$

where the coefficients A_j, B_{jk}, and C_j are C^∞ functions on Weyl chambers.

By condition (D), the third order terms of $[\operatorname{rad}(P), \operatorname{rad}(\Delta_M)] = 0$, and we get the equations

$$\frac{\partial}{\partial t_j} A_j = \frac{1}{2} \sum_{\alpha \neq j} ((a_\alpha)_{t_\alpha t_\alpha} + a_\alpha (a_\alpha)_{t_j}), \tag{5-4}$$

$$\frac{\partial}{\partial t_j} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_j (a_j)_{t_k} - 2(A_j)_{t_k}, \tag{5-5}$$

$$\frac{\partial}{\partial t_k} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_k (a_k)_{t_j} - 2(A_k)_{t_j}, \tag{5-6}$$

where we put $a_j = \Omega_{t_j}/\Omega$.

We take

$$\begin{aligned} A_j = &-(n+1-2r) \sum_{\alpha \neq j} \cot t_j \frac{\sin 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j} \\ &+ 2 \sum_{\alpha \neq j} \left(\frac{\cos 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j} - \frac{\sin^2 2t_\alpha}{(\cos 2t_\alpha - \cos 2t_j)^2} \right) \\ &+ \sum_{\alpha \neq j} \frac{\sin^2 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j} \\ &- 2 \sum_{\substack{\alpha < \beta \\ \alpha, \beta \neq j}} \left\{ 1 + \frac{\sin^2 2t_\alpha}{(\cos 2t_\alpha - \cos 2t_j)(\cos 2t_\beta - \cos 2t_j)} \right\}, \end{aligned} \tag{5.7}$$

$$B_{jk} = \frac{3}{2} a_{j,t_k} + a_j a_k. \tag{5.8}$$

After a tedius but straight forward computation, we find that the functions (5.7) and (5.8) satisfy the equations (5.4), (5.5), and (5.6).

We get by the condition (B),

$$C_j = \frac{1}{\Omega} (A_j \Omega)_{t_j} + \frac{1}{2\Omega} \sum_{j<k} \{(B_{jk}\Omega)_{t_k} + (B_{jk}\Omega)_{t_j}\} - \frac{1}{2\Omega} \sum_{j \neq k} \Omega_{t_j t_k t_k} \tag{5.9}$$

We define an operator Q_1 by the right hand side of the equation (5.3), where the coefficients A_j, B_{jk}, and C_j are given by the functions (5.7), (5.8), and (5.9), respectively. A differential operator $Q_2 := \mathrm{rad}(P) - Q_1$ is a second order differential operator and satisfies the conditions (B), (C), and (D). We will prove that the operator Q_2 can be written as $c\ \mathrm{rad}(\Delta_M)$ for a suitable constant c.

We define a subgroup W_0 of $W(G,K)$ by the set of all maps s in (4.1) such that $\varepsilon_1\ \varepsilon_2 \cdots \varepsilon_r = 1$. If $n+1 > 2r$, W_0 is strictly contained in $W(G,K)$, and if $n+1 = 2r$, W_0 is identical with $W(G,K)$.

Lemma 5.2. *We assume that* $(n,r) \neq (3,2)$. *If a second order differential operator* Q *satisfies the conditions (B) and (D), and if* Q *is* W_0*-invariant, then* $Q = c\ \mathrm{rad}(\Delta_M)$ *for some constant* c.

Proof. We put

$$Q := \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial t_j^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j}$$

By the condition (D), the third order terms of $[Q, \mathrm{rad}(\Delta_M)]$ vanish, which means

$$A_{j,t_j} = 0, \qquad (1 \leq j \leq r); \tag{5.10}$$

$$A_{k,t_j} + B_{jk,t_k} = 0, \quad A_{j,t_k} + B_{jk,t_j} = 0, \qquad (j<k); \tag{5.11}$$

$$B_{ij,t_k} + B_{jk,t_i} + B_{ik,t_j} = 0, \qquad (1 \leq i < j < k \leq r). \tag{5.12}$$

By the equations (5.10) $\sim$ (5.12) and the assumptions that Q is W_0-invariant and that $(n,r) \neq (3,2)$, the coefficients A_j and B_{jk} are polynomials of the form

$$A_j = \delta_1 \sum_{k \neq j} t_k^2 + \delta_2, \tag{5.13}$$

$$B_{jk} = -2\delta_1 t_j t_k, \tag{5.14}$$

where δ_1 and δ_2 are some constants.

Using the condition (B), we have

$$C_j = \frac{1}{\Omega}(A_j\Omega)_{t_j} + \frac{1}{2\Omega}\sum_{j<k}(B_{jk}\Omega)_{t_k} + \frac{1}{2\Omega}\sum_{k<j}(B_{kj}\Omega)_{t_k} \tag{5.15}$$

If $\delta_1 = 0$, then the coefficient $B_{jk} = 0$, and the coefficient $C_j = \delta_2 \Omega_{t_j}/\Omega$ by (5.15). Therefore, we obtain $Q = -(n-1)\delta_2\ \mathrm{rad}(\Delta_M)$, and the lemma holds.

Now, we suppose that $\delta_1 \neq 0$ In particular, we may suppose that $\delta_1 = 1$ and $\delta_2 = 0$. By the condition (D), the first order terms of $[Q, \mathrm{rad}(\Delta_M)]$ vanish. Then we have

$$Qa_1 = -(n-1)\,\mathrm{rad}(\Delta_M)C_1), \tag{5.16}$$

where $a_1 = \Omega_{t_1}/\Omega$.

We extend the both sides of (5.16) to C as meromorphic functions of $t_1 = \mu_1 + \sqrt{-1}\nu_1$.

By the formula (5.1), we have

$$a_1 = (n+1-2r)\frac{\cos t_1}{\sin t_1} + \sum_{j=2}^{r} \frac{-2\sin 2t_1}{\cos 2t_1 - \cos 2t_j} \tag{5.17}$$

Let $\nu_1 \to +\infty$, then $a_{1,t_j} \to 0$, $a_{1,t_j t_k} \to 0$ (rapidly decreasing), and $a_1 = O(1)$. The same fact holds for a_j ($j = 2, \ldots, r$). Thus, $Qa_1 \to 0$ (rapidly decreasing). Therefore, we get $\mathrm{rad}(\Delta_M)C_1 \to 0$ (rapidly decreasing) by (5.16). However, when ν_1 tends to $+\infty$, we have

$$\begin{aligned} -(n-1)\,\mathrm{rad}(\Delta_M)C_1 &= \frac{1}{2}\sum_{j,k=2}^{r}\left(\frac{\partial^2}{\partial t_k{}^2} + a_k\frac{\partial}{\partial t_k}\right)(B_{1j}a_j + B_{1j,t_j}) + O(1) \\ &= -t_1\sum_{k=2}^{r} a_k{}^2 + O(1). \end{aligned}$$

(In the above computation, we have used (5.13), (5.14) and the fact that $a_j = O(1)$ and the derivatives of $a_j \to 0$ as $\nu_1 \to \infty$.) Therefore, we have $\mathrm{rad}(\Delta_M)C_1 \to \infty$, for suitable $t_2, \ldots, t_r$, and μ_1. It is a contradiction. ∎

Remark 5.3. When $n = 3$ and $r = 2$, a differential operator

$$Q = \frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\Omega_{t_2}}{2\Omega}\frac{\partial}{\partial t_1} + \frac{\Omega_{t_1}}{2\Omega}\frac{\partial}{\partial t_2}, \tag{5.18}$$

satisfies the conditions (B), (C), and (D). It is obvious that Q is linearly independent of $\mathrm{rad}(\Delta_M)$. Therefore, it is easily checked that this operator Q is the radial part of $P = L_{34,12}$.

By the above argument, we get the following proposition.

Proposition 5.4. *The differential operator* $\mathrm{rad}(P)$ *can be written in the form*

$$\mathrm{rad}(P) = Q_1 + c(n-1)\,\mathrm{rad}(\Delta_M),$$

for some constant c.

§6. Proof of Theorem 1.1.

We calculate the eigenvalue of P on $V(m_1, \ldots, m_r)$ to prove Theorem 1.1.

Let $a(m_1, \ldots, m_r)$ be the eigenvalue of P on $V(m_1, \ldots, m_r)$ and $\phi_{(m_1,\ldots,m_r)}$ the zonal spherical function which belongs to $V(m_1, \ldots, m_r)$. We denote by $u_{(m_1,\ldots,m_r)}$ the restriction of $\phi_{(m_1,\ldots,m_r)}$ to the Weyl chamber $\mathcal{A}^+$ Since the procedure is almost the same as that in [10], we omit the proofs of the following lemmas.

Lemma 6.1. ([12], Theorem 8.1.) *The function $u_{(m_1,\ldots,m_r)}$ has a Fourier series expansion on A^+ of the form*

$$u_{(m_1,\ldots,m_r)}(t_1,\ldots,t_r) \tag{6.1}$$
$$= \sum_{\substack{\lambda \le m_1M_1+\cdots+m_rM_r \\ \lambda\in Z(G,K),\text{finite sum}}} \eta_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r),$$

where $\eta_{m_1M_1+\cdots+m_rM_r} > 0$.

Let f_1 and f_2 be Fourier series of the form

$$f_1 = \sum_{\lambda\le\Lambda_1,\lambda\in Z(G,K)} \zeta_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r)$$
$$f_2 = \sum_{\lambda\le\Lambda_2,\lambda\in Z(G,K)} \tilde{\zeta}_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r)$$

We denote $f_1 \sim f_2$ when $\Lambda_1 = \Lambda_2 (>0)$ and $\zeta_{\Lambda_1} = \tilde{\zeta}_{\Lambda_2} (\neq 0)$.

Lemma 6.2.

$$\Omega_{t_j} \sim \sqrt{-1}(n+1-2j)\Omega, \tag{6.2}$$
$$A_j\Omega^2 \sim -(n-j)(j-1)\Omega^2, \tag{6.3}$$
$$B_{jk}\Omega^2 \sim -(n+1-2j)(n+1-2k)\Omega^2, \tag{6.4}$$
$$C_j\Omega^3 \sim -\sqrt{-1}(n-j)(j-1)(n+1-2j)\Omega^3, \tag{6.5}$$

where the functions A_j, B_{jk}, and C_j are given by (5.5), (5.6), *and* (5.7), *respectively.*

Theorem 6.3. *Unless $n=3$ and $r=2$, the eigenvalue $a(m_1,\ldots,m_r)$ of P on $V(m_1,\ldots,m_r)$ is given by the formulae*

$$a(m_1,\ldots,m_r) = \sum_{1\le j<k\le r} l_jl_k(l_j+n+1-2j)(l_k+n+1-2k)$$
$$+\sum_{j=2}^{r}(j-1)(n-j)l_j(l_j+n+1-2j),$$

where l_j is given as follows.

$$(n+1>2r+1):\quad l_j = 2(m_j+\cdots+m_{r-1})+m_r$$
$$(n+1=2r, or\quad 2r+1):\quad l_j = 2(m_j+\cdots+m_r).$$

Proof. By definition, we have

$$P\phi_{(m_1,\dots,m_r)} = a(m_1,\dots,m_r)\phi_{(m_1,\dots,m_r)}.$$

We restrict both sides to the Weyl chamber $\mathcal{A}^+$, and then we have

$$\mathrm{rad}(P)u_{(m_1,\dots,m_r)} = a(m_1,\dots,m_r)u_{(m_1,\dots,m_r)}.$$

By Proposition 5.3, Lemma 6.1 and Lemma 6.2, it follows that

$$\begin{aligned}
&\Omega^3\,\mathrm{rad}(P)u_{(m_1,\dots,m_r)}\\
&= \Omega^3\{Q_1 + c(n-1)\,\mathrm{rad}(\Delta_M)\}u_{(m_1,\dots,m_r)}\\
&\sim \{\sum_{1\le j<k\le r} l_j l_k(l_j+n+1-2j)(l_k+n+1-2k)\\
&\quad+\sum_{j=2}^{r}(j-1)(n-j)l_j(l_j+n+1-2j)\}\Omega^3 u_{(m_1,\dots,m_r)}\\
&\quad+c\sum_{j=1}^{r} l_j(l_j+n+1-2j)\Omega^3 u_{(m_1,\dots,m_r)}.
\end{aligned} \tag{6.6}$$

Thus, we have

$$\begin{aligned}
a(m_1,\dots,m_r) = &\sum_{1\le j<k\le r} l_j l_k(l_j+n+1-2j)(l_k+n+1-2k)\\
&+\sum_{j=2}^{r}(j-1)(n-j)l_j(l_j+n+1-2j)\\
&+c\sum_{j=1}^{r} l_j(l_j+n+1-2j).
\end{aligned} \tag{6.7}$$

Here, by Proposition 2.1 and Lemma 4.1, we have $a(2m,0,\dots,0)=0$. Therefore, we get $c=0$, which completes the proof. ∎

Remark 6.4. For the case $l=1$, the eigenvalue of P can be also computed by using the method of radial part.

The following corollary is easily verified.

Corollary 6.5. *$V(m_1,\dots,m_r)$ is contained in* $\mathrm{Ker}\,P$ *if and only if $m_2 = = m_r = 0$.*

Proof of Theorem 1.1. Our proof of Theorem 1.1 is almost the same as that of Theorem 1.2 in [10].

Let $V := \bigoplus_{m=0}^{\infty} V(m,0,\dots,0)$ and $\tilde V := \bigoplus_{m=0}^{\infty} V_{2m}$. Then we have $R:\tilde V\to V$ and $S: V\to\tilde V$ Moreover, we have $SR = Id$ on $\tilde V$ and $RS = Id$ on V by Proposition 3.2 and Lemma 4.1.

By Corollary 6.5, V is dense in $\mathrm{Ker}\,P$ in C^∞-topology. Since $S: C^\infty(M)\to C^\infty_{even}(\mathbf{S}^n)$ is continuous, we have $RS = Id$ on $\mathrm{Ker}\,P$ This proves Theorem 1.1. ∎

Remark 6.6. The differential operator P is of the least degree in all the invariant differential operators on $\widetilde{Gr}(l,n;\mathbf{R})$ that characterize the range of R. It follows from the fact that the principal symbol $F(X)$ of P, which we defined in Section 1, is of the least degree in all the Ad-K-invariant polynomials on $\mathfrak{m}$ except for the principal symbol of the Laplacian.

§7. Radon transforms on $\mathbf{P}^n\mathbf{R}$.

The set of all projective l-dimensional plane in $\mathbf{P}^n\mathbf{R}$ is the real Grassmann manifold $Gr(l,n;\mathbf{R})$, which is a compact symmetric space $O(n+1)/O(l+1)\times O(n-l)$ of rank $\min\{l+1,n-l\}$. We define a Radon transform $\mathcal{R}: C^\infty(\mathbf{P}^n\mathbf{R})\to C^\infty(Gr(l,n;\mathbf{R}))$ as follows.

$$\mathcal{R}f(\eta)=\frac{1}{\mathrm{Vol}(\mathbf{P}^l\mathbf{R})}\int_{x\in\eta}f(x)dv_\eta(x),$$

where $dv_\eta(x)$ is the canonical measure on $\eta(\subset\mathbf{P}^n\mathbf{R})$.

Since we can identify $C^\infty_{even}(\mathbf{S}^n)$ with $C^\infty(\mathbf{P}^n\mathbf{R})$, we have

$$Rf(+\eta)=Rf(-\eta)=\mathcal{R}f(\eta)\quad\text{for } f\in C^\infty(\mathbf{P}^n\mathbf{R}) \text{ and } \eta\in Gr(l,n;\mathbf{R}),$$

where $+\eta$ and $-\eta$ denote orientations of η.

We defined the invariant differential operator P on $\widetilde{Gr}(l,n;\mathbf{R})$ in Section 2, but we can easily check that P is also well-defined as an invariant differential operator on $Gr(l,n;\mathbf{R})$. Therefore, we obtain the following theorem from Theorem 1.1.

Theorem 7.1. *The range of the Radon transform $\mathcal{R}$ on $\mathbf{P}^n\mathbf{R}$ is identical with* $\operatorname{Ker} P$

Remark 7.2. For technical reasons, we first gave the proof of the range theorem in the case of $\mathbf{S}^n$ In fact, the compact oriented real Grassmann manifold $\widetilde{Gr}(l,n;\mathbf{R})$ is a simply connected symmetric space of type BD I and its irreducible representations are well-known.

Chapter 3.

Range characterization of Radon transforms on quaternionic projective spaces

§0. On the main theorem.

In this chapter, we consider the range characterization of the Radon transforms on the quaternionic projective spaces.

We denote by $Gr(l,n;\mathbf{H})$ the set of all projective l-planes in $\mathbf{P}^n\mathbf{H}$, which is a quaternionic Grassmann manifold and is a compact symmetric space of rank $\min\{l+1, n-l\}$. For simplicity, we sometimes denote $Gr(l,n;\mathrm{H})$ by M, and as in Chapter 1 or in Chapter 2, we put $r = \operatorname{rank} M = \min\{l+1, n-l\}$.

We define a Radon transform $R : C^\infty(\mathbf{P}^n\mathbf{H}) \to C^\infty(M)$ by

$$Rf(\xi) = \frac{1}{\mathrm{Vol}(\mathbf{P}^l\mathbf{H})} \int_{x\in\xi} f(x)dv_\xi(x), \quad \xi \in M, \quad f \in C^\infty(\mathbf{P}^n\mathbf{H}),$$

where $dv_\xi(x)$ denotes the measure on ξ ($\subset \mathbf{P}^n\mathbf{H}$) induced by the canonical measure on $\mathbf{P}^n\mathbf{H}$.

In this case, the same result as in Chapter 1 or in Chapter 2 holds. Our answer is the following.

Theorem. *Under the assumption $1 \leq l \leq n-2$, there is a fourth order invariant differential operator P on a quaternionic Grassmann manifold $Gr(l,n;\mathrm{H})$ such that the range $\operatorname{Im} R$ of the projective l-plane Radon transform $R = R_l$ on $\mathbf{P}^n\mathbf{H}$ is identical with the kernel of P, that is,* $\operatorname{Im} R = \operatorname{Ker} P$

The basic tools are the same as those in Chapter 1 or in Chapter 2. We use the inversion formula and the method of radial part, and we prove the above theorem in the same way as in Chapter 1 or in Chapter 2. As we mentioned before, we can not represent the range-characterizing operator P in a similar form as in Chapter 1 or in Chapter 2, because of the non-commutativity of the quaternion field $\mathbf{H}$. However, we find that the radial part of the range-characterizing operator is of the similar form as in Chapter 1 or in Chapter 2 and is an ultrahyperbolic type differential operator on Weyl chambers. Therefore, in this sense, we can also say that the ranges of R can be characterized by ultrahyperbolic type differential operators.

The notation in this chapter is the same as in Chapter 1 or in Chapter 2.

§1. Representation of Gr(l, n; H).

We denote by $Gr(l, n; \mathbf{H})$ the quaternionic Grassmann manifold of all the projective l-planes of the quaternionic projective space $\mathbf{P}^n\mathbf{H}$. Then $\mathbf{P}^n\mathbf{H}$ and $Gr(l, n; \mathbf{H})$ are compact symmetric spaces $Sp(n+1)/Sp(1) \times Sp(n)$ of rank 1 and $Sp(n+1)/Sp(l+1) \times Sp(n-l)$ of rank $\min\{l+1, n-l\}$, respectively. Here $Sp(m)$ denotes a compact Lie group defined by

$$Sp(m) = \{g \in U(2m); {}^tgJ_mg = J_m\},$$

where $J_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$ and I_m denotes the $m \times m$ identity matrix. The imbedding of $Sp(l+1) \times Sp(n-l)$ into $Sp(n+1)$ is given by the mapping

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix},$$

where A_1, B_1, C_1, and D_1 are $(l+1) \times (l+1)$ matrices, and A_2, B_2, C_2, and D_2 are $(n-l) \times (n-l)$ matrices.

For simplicity, we put $G = Sp(n+1)$ and $K_l = Sp(l+1) \times Sp(n-l)$ in this chapter.

Let $\mathfrak{g}$ and $\mathfrak{k}_l$ be the Lie algebras of G and K_l, respectively. Then we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z} \end{pmatrix}; \begin{array}{c} Z_1, Z_2 \,\text{comlex}\, (n+1) \times (n+1) \,\text{matrix} \\ Z_1 \,\text{skew hermitian},\ Z_2 \,\text{symmetric} \end{array} \right\}$$

$$\mathfrak{k}_l = \left\{ \begin{pmatrix} X_1 & 0 & X_2 & 0 \\ 0 & Y_1 & 0 & Y_2 \\ -\bar{X}_2 & 0 & \bar{X}_1 & 0 \\ 0 & -\bar{Y}_2 & 0 & \bar{Y}_1 \end{pmatrix} \in \mathfrak{g}; \begin{array}{ll} X_1, Y_1 & (l+1) \times (l+1) \,\text{matrix} \\ X_2, Y_2 & (n-l) \times (n-l) \,\text{matrix} \\ X_1, Y_1 & \text{skew hermitian} \\ X_2, Y_2 & \text{symmetric} \end{array} \right\}$$

Let $\mathfrak{g} = \mathfrak{k}_l \oplus \mathfrak{m}_l$ be the Cartan decomposition, where $\mathfrak{m}_l$ is the set of all the matrices of the form

$$X = \begin{pmatrix} 0 & -Z^* & 0 & {}^tW \\ Z & 0 & W & 0 \\ 0 & -W^* & 0 & -{}^tZ \\ -\bar{W} & 0 & \bar{Z} & 0 \end{pmatrix}, \tag{1.1}$$

where Z and W are arbitrary complex $(l+1) \times (n-l)$ matrices. We define a matrix $H(t) = H(t_1, \ldots, t_r) \in \mathfrak{m}_l$ by taking $W = 0$ and $Z = \sqrt{-1}(t_1E_{11} + \cdots + t_rE_{rr})$ in (1.1). Here $t = (t_1, \ldots, t_r) \in \mathbf{R}^r$, and E_{ii} denotes the $(n-l) \times (l+1)$ matrix whose (i, i) entry is 1 and whose other entries are 0.

Let $\mathfrak{a}$ be the set of all the matrices $H(t)$, and then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}_l$. We identify this subspace $\mathfrak{a}$ of $\mathfrak{m}_l$ with $\mathbf{R}^r$ by the mapping $H(t) \mapsto t$.

Let $(\ ,\ )$ denote an invariant inner product on $\mathfrak{g}$ defined by

$$(A, B) = -2(n+2)\,\mathrm{tr}(AB) \quad A, B \in \mathfrak{g},$$

which is a minus signed Killing form on $\mathfrak{g}$.

We choose a Killing form metric on G, which induces metrics on K_l, $\mathrm{P}^n\mathrm{H}\,(= G/K_0)$ $Gr(l, n; \mathbf{H})$, $(= G/K_l)$, respectively.

For $\alpha \in \mathfrak{a}$, let $\mathfrak{g}_\alpha := \{\, X \in \mathfrak{g}^{\mathbf{C}}; [H, X] = \sqrt{-1}(\alpha, H)X$ for all $H \in \mathfrak{a}\,\}$, then α is called a root of $(\mathfrak{g}, \mathfrak{a})$, when $\mathfrak{g}_\alpha \neq \{0\}$. We put $m_\alpha = \dim_{\mathbf{C}}\mathfrak{g}_\alpha$, and call it the multiplicity of α.

We put $H_i = H(0, \ldots, \overset{(i)}{1}, \ldots, 0)$ $(1 \leq i \leq r)$. Then the roots of $(\mathfrak{g}, \mathfrak{a})$ and their multiplicities are given by the following table.

α	m_α	
$\pm\frac{1}{4(n+1)}H_j$	3	$(1 \leq j \leq r)$,
$\pm\frac{1}{8(n+1)}H_j$	$4(n+1-2r)$	$(1 \leq j \leq r)$,
$\pm\frac{1}{8(n+1)}(H_j \pm H_k)$	4	$(1 \leq j < k \leq r)$.

We fix a lexicographical order $<$ on $\mathfrak{a}$ such that $H_1 > \cdots > H_r > 0$.

The simple roots α_j $(1 \leq j \leq r)$ are given by the following table.

$$(n+1 > 2r): \qquad \alpha_j = \frac{1}{8(n+2)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-1),$$
$$\alpha_r = \frac{1}{8(n+2)}H_r.$$
$$(n+1 = 2r): \qquad \alpha_j = \frac{1}{8(n+2)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-1),$$
$$\alpha_r = \frac{1}{4(n+2)}H_r.$$

Let M_j $(1 \leq j \leq r)$ be the fundamental weights of G/K_l corresponding to the simple roots α_j $(1 \leq j \leq r)$. Then M_j $(1 \leq j \leq r)$ are given by the following table.

$$(n+1 > 2r): \qquad M_j = \frac{1}{4(n+2)}\sum_{k=1}^{j} H_k \quad (1 \leq j \leq r),$$
$$(n+1 = 2r): \qquad M_j = \frac{1}{4(n+2)}\sum_{k=1}^{j} H_k,$$
$$M_r = \frac{1}{2(n+2)}\sum_{k=1}^{r} H_k.$$

The Weyl group $W(G, K_l)$ of (G, K_l) is the set of all maps $s : \mathfrak{a} \to \mathfrak{a}$ such that

$$s : (t_1, \ldots, t_r) \mapsto (\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_r t_{\sigma(r)}), \quad \varepsilon_j = \pm 1, \sigma \in \mathfrak{S}_r,$$

where $\mathfrak{S}_r$ denotes the symmetric group of degree r

Let $Z(G, K_l)$ be the weight lattice, that is, $Z(G, K_l) = \{ \frac{1}{4(n+2)}(\mu_1 H_1 + \cdots + \mu_r H_r); \mu_1, \ldots, \mu_r \in \mathbf{Z} \}$. A highest weight of a spherical representation of (G, K_l) is of the form $m_1 M_1 + \cdots + m_r M_r$, where $m_1, \ldots, m_r$ are non-negative integers. Let $V(m_1, \ldots, m_r)$ denote the eigenspace of the Laplacian Δ_{G/K_l} on $G/K_l (= Gr(l, n; \mathbf{H}))$ that is an irreducible representation space with the highest weight $m_1 M_1 + \cdots + m_r M_r$. In particular, we denote by $V(m) = V_m$ the m-th eigenspace of $\Delta_{G/K_0} = \Delta_{\mathbf{P}^n\mathbf{H}}$.

§2. **Invariant differential operators on** $\mathrm{Gr}(\mathrm{l}, \mathrm{n}; \mathbf{H})$.

In this section, we describe the invariant differential operators on the symmetric space $Gr(l, n; \mathbf{H})$.

Let F_j $(1 \leq j \leq 2r)$ be Ad-K-invariant polynomials on $\mathfrak{m}_l$ defined by

$$\det(\lambda I + X) = \lambda^{2(n+1)} + F_1(X)\lambda^{2n} + F_2(X)\lambda^{2(n-1)} + \cdots + F_{2r}(X)\lambda^{2(n+1-2r)}$$

We define invariant differential operators P_j $(1 \leq j \leq 2r)$ of order $2j$ on G/K_l as follows.

$$P_j f(g) = F_j(\partial X) f(g \exp X)|_{X=0} \quad \text{for } f \in C^\infty(G, K_l).$$

Here X $(\in \mathfrak{m}_l)$ is the matrix of the form,

$$X = \begin{pmatrix} 0 & -Z^* & 0 & {}^tW \\ Z & 0 & W & 0 \\ 0 & -W^* & 0 & -{}^tZ \\ -\bar{W} & 0 & \bar{Z} & 0 \end{pmatrix},$$

where we denote the (i, j) entries of the $(l+1) \times (n-l)$ matrix Z, W, $\bar{Z}$, and $\bar{W}$ by z_{ij}, w_{ij}, $\bar{z}_{ij}$, and $\bar{w}_{ij}$, respectively, and ∂X denotes the matrix valued differential operator which is obtained by replacing z_{ij}, w_{ij}, $\bar{z}_{ij}$, and $\bar{w}_{ij}$ for $\frac{\partial}{\partial z_{ij}}$, $\frac{\partial}{\partial w_{ij}}$, $\frac{\partial}{\partial \bar{z}_{ij}}$, and $\frac{\partial}{\partial \bar{w}_{ij}}$, respectively, in X.

The matrix ∂X is represented formally as

$$\partial X = \begin{pmatrix} 0 & -\partial Z^* & 0 & \partial^t W \\ \partial Z & 0 & \partial W & 0 \\ 0 & -\partial W^* & 0 & -\partial^t Z \\ -\partial \bar{W} & 0 & \partial \bar{Z} & 0 \end{pmatrix}$$

Remark 2.1. The following facts (1) is well-known and the facts (2) and (3) are easily checked.

(1) $P_1, \ldots, P_{2r}$ are commutative with each other.

(2) The operator P_j is formally self-adjoint with respect to the L^2 inner product on G/K_l.

(3) The algebra $\mathcal{L}(G/K_l)$ is generated by $P_1, \ldots, P_r$, where $\mathcal{L}(G/K_l)$ denotes the algebra of all the invariant differential operators on G/K_l.

In this chapter, two operators P_1 and P_2 play an important role, that is, the range-characterizing operator will be constructed by using P_1 and P_2.

§3. The inversion formula.

We construct a continuous linear map $S : C^\infty(Gr(l,n;\mathbf{H})) \to C^\infty(\mathbf{P}^n\mathbf{H})$ such that $SR = Id$, where Id denotes the identity map.

We define another type of Radon transform $\mathcal{F}_l$ which is a mapping from $C^\infty(Gr(l,n;\mathbf{H}))$ to $C^\infty(Gr(n-1,n;\mathbf{H}))$, that is, $\mathcal{F}_l : C^\infty(G/K_l) \to C^\infty(G/K_{n-1})$ as follows.

$$(\mathcal{F}_l f)(g) = \frac{1}{\mathrm{Vol}(K_{n-1})} \int_{k \in K_{n-1}} f(gk)dk, \quad f \in C^\infty(G, K_l).$$

In particular, we denote $\mathcal{F}_0 = \mathcal{F}$ Geometrically, $Gr(n-1,n;\mathbf{H})$ $(= G/K_{n-1})$ means the manifold of all the antipodal manifolds in $\mathbf{P}^n\mathbf{H}$ and $\mathcal{F} : C^\infty(\mathbf{P}^n\mathbf{H}) \to C^\infty(Gr(n-1,n;\mathbf{H}))$ means the Radon transform defined by integrating functions on $\mathbf{P}^n\mathbf{H}$ over antipodal manifolds and taking mean values.

Similarly, we define a dual Radon transform $\tilde{\mathcal{F}} : C^\infty(G/K_{n-1}) \to C^\infty(G/K_0) = C^\infty(\mathbf{P}^n\mathbf{H})$ of $\mathcal{F}$ by

$$(\tilde{\mathcal{F}} f)(g) = \frac{1}{\mathrm{Vol}(K_{n-1})} \int_{k \in K_{n-1}} f(gk)dk, \quad f \in C^\infty(G, K_{n-1}).$$

Then the following theorem holds.

Theorem 3.1. (Helgason [7], Ch. 1, Theorem 4.11.) *We have the inversion formula,*

$$\Phi(\Delta_{\mathbf{P}^n\mathbf{H}})\tilde{\mathcal{F}}\mathcal{F} = Id.$$

Here $\Phi(t)$ is a polynomial of t of degree $2(n-1)$ defined by

$$\Phi(t) = c_n \prod_{j=1}^{2(n-1)} \left(t + \frac{(2n-j)(j+1)}{4(n+2)} \right)$$

$$\textit{where} \quad c_n = 4(n+2)^{2(n-1)} \prod_{j=1}^{2(n-1)} \{(2n-j)(j+1)\}^{-1}$$

We prove the following proposition, using the above Helgason's inversion formula.

Proposition 3.2. *For the Radon transform $R = R_l$, there exists an inversion map $S = S_l : C^\infty(Gr(l,n;\mathbf{H}) \to C^\infty(\mathbf{P}^n\mathbf{H})$ such that $S_l R_l = Id$.*

Proof. We can easily check that $\mathcal{F}_l R_l = \mathcal{F}$ Therefore, if we put

$$S_l = \Phi(\Delta_{\mathbf{P}^n\mathbf{H}})\bar{\mathcal{F}}\mathcal{F}_l, \tag{3.1}$$

we get $S_l R_l = Id$ by Theorem 3.1. ∎

Corollary 3.3. *The Radon transform $R = R_l$ isomorphically maps the subspace V_m of $C^\infty(\mathbf{P}^n\mathbf{H})$ to the subspace $V(m,0,\dots,0)$ of $C^\infty(Gr(l,n;\mathbf{H})$.*

Proof. Since the proof is almost the same as that of Lemma 6.4 in Chapter 1 or that of Lemma 4.1 in Chapter 2, we give only the sketch of the proof. The Radon transform R is G-equivariant and, by Proposition 3.2, R is injective. On the other hand, it is easily checked that the highest weight of V_m corresponds to that of $V(m,0,\dots,0)$. Therefore, the assertion is verified.

§4. Radial part of invariant differential operators on $\mathbf{G/K_l}$.

We define a density function Ω on $\mathfrak{a}$ by

$$\Omega(t) = \Big|\prod_{\alpha:\text{positive root}} 2\sin(\alpha, H(t))^{m_\alpha}\Big|$$

Then we have

$$\begin{aligned} \Omega &= \sigma\omega^4, \\ \text{where}\quad \sigma &= 2^{r(2n-2r+3)}\Big|\prod_{j=1}^{r}\sin^3 2t_j \sin^{2(n-r+1)} t_j\Big|, \\ \omega &= 2^{\frac{1}{2}r(r-1)}\prod_{j<k}(\cos 2t_j - \cos 2t_k). \end{aligned}$$

We define a Weyl chamber $\mathcal{A}$ by the set $\{t \in \mathbf{R}^r; \Omega(t) \neq 0\}$.

To each invariant differential operator D on G/K, there corresponds a unique differential operator on Weyl chambers which is invariant under the action of the Weyl group $W(G,K)$. This operator is called a radial part of D, and we denote it by $\mathrm{rad}(D)$.

The radial part of the Laplacian Δ_{G/K_l} on G/K_l is given by

Lemma 4.1.

$$\mathrm{rad}(\Delta_{G/K}) = -\frac{1}{8(n+2)}\sum_{j=1}^{r}\left(\frac{\partial^2}{\partial {t_j}^2} + \frac{\Omega_{t_j}}{\Omega}\frac{\partial}{\partial t_j}\right)$$

where Ω_{t_j} denotes the differentiation of Ω by t_j.

For the proof, see [12], Theorem 10.4.

We define an invariant differential operator L_c on $M = G/K$ by

$$L_c = \frac{1}{4}P_2 - \frac{1}{8}P_1{}^2 + c\Delta_M, \tag{4.1}$$

where c is a real constant.

We denote the principal symbol of L_c by F Then F is given by

$$\begin{aligned} F &= \frac{1}{16}\left(\frac{1}{4}F_2 - \frac{1}{8}F_1{}^2\right) \\ &= \frac{1}{128}(2F_2 - F_1{}^2). \end{aligned}$$

From now, we calculate the radial part of the invariant differential operator L_c.

We consider the following conditions (A), (B), (C), and (D) on a differential operator Q on $\mathbf{R}^r$ that is regular in all Weyl chambers.

(A) $Q = \sum_{j<k} \frac{\partial^2}{\partial t_j{}^2}\frac{\partial^2}{\partial t_k{}^2}$ + lower order terms.

(B) Q is formally self-adjoint with respect to the density Ωdt.

(C) Q is $W(G,K)$-invariant.

(D) $[Q, \mathrm{rad}(\Delta_M)] := Q\,\mathrm{rad}(\Delta_M) - \mathrm{rad}(\Delta_M)Q = 0$.

Lemma 4.2. $16\,\mathrm{rad}(L_c)$ *satisfies the above four conditions* (A), (B), (C), *and* (D).

Proof. By the definition of F_j $(1 \le j \le 2r)$,

$$\det(\lambda I + H(t)) = \\ \lambda^{2(n+1)} + F_1(H(t))\lambda^{2n} + F_2(H(t))\lambda^{2(n-1)} + \cdots + F_{2r}(H(t))\lambda^{2(n+1-2r)}$$

Thus, we have

$$\begin{aligned} F_1(H(t)) &= 2\sum_{j=1}^{r} t_j{}^2 \\ F_2(H(t)) &= \sum_{j=1}^{r} t_j{}^4 + 4\sum_{j<k} t_j{}^2 t_k{}^2 \end{aligned}$$

Therefore, the restriction of F to $\mathfrak{a}$ is given by

$$\begin{aligned} 16F(H(t)) &= \frac{1}{4}F_2(H(t)) - \frac{1}{8}F_1(H(t))^2 \\ &= \sum_{j<k} t_j{}^2 t_k{}^2 \end{aligned}$$

Since the principal symbol of $\mathrm{rad}(P)$ is given by $F(H(t))$, the above equation proves that the condition (A) holds for $Q = 16\,\mathrm{rad}(L_c)$.

The condition (B) follows from the fact (1) in Remark 1.1.

The conditions (C) and (D) are well-known facts. For detail, see [10]. ∎

We calculate the explicit form of the differential operator Q.

By the conditions (A) and (B), we get

$$\text{the third order terms of } \mathrm{rad}(P) = \sum_{j\neq k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial {t_j}^2 \partial t_k}$$

Thus, we can put

$$\begin{aligned} Q = &\sum_{j<k} \frac{\partial^4}{\partial {t_j}^2 \partial {t_k}^2} + \sum_{j\neq k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial {t_j}^2 \partial t_k} \\ &+ \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial {t_j}^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j}, \end{aligned} \tag{4.4}$$

where the coefficients A_j, B_{jk}, and C_j are C^∞ functions on Weyl chambers.

By condition (D), the third order terms of $[\mathrm{rad}(P), \mathrm{rad}(\Delta_M)] = 0$, and we get the equations

$$\frac{\partial}{\partial t_j} A_j = \frac{1}{2} \sum_{\alpha\neq j} ((a_\alpha)_{t_\alpha t_\alpha} + a_\alpha (a_\alpha)_{t_j}), \tag{4.3}$$

$$\frac{\partial}{\partial t_j} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_j (a_j)_{t_k} - 2(A_j)_{t_k}, \tag{4.4}$$

$$\frac{\partial}{\partial t_k} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_k (a_k)_{t_j} - 2(A_k)_{t_j}, \tag{4.5}$$

where we put $a_j = \Omega_{t_j}/\Omega$.

Then one of the solutions of the equations (4.3–5) is given by

$$\begin{aligned} A_j = &\sum_{\alpha\neq j} \left\{ \chi(t_\alpha) \frac{(\omega_{j\alpha})_{t_\alpha}}{\omega_{j\alpha}} + 2 \left(\frac{(\omega_{j\alpha})_{t_\alpha}}{\omega_{j\alpha}} \right)_{t_\alpha} + 2 \left(\frac{(\omega_{j\alpha})_{t_\alpha}}{\omega_{j\alpha}} \right)^2 \right\} \\ &- 8 \sum_{\substack{\alpha<\beta \\ \alpha,\beta\neq j}} \left\{ 4 + \frac{(\omega_{j\alpha})_{t_j}}{\omega_{j\alpha}} \frac{(\omega_{j\beta})_{t_j}}{\omega_{j\beta}} \right\}, \end{aligned} \tag{4.6}$$

$$B_{jk} = \frac{1}{\Omega^2} \{3\Omega\, \Omega_{t_j t_k} - 2\Omega_{t_j} \Omega_{t_k}\}, \tag{4.7}$$

where $\chi(t_j) = \sigma_{t_j}/\sigma = 6\cot 2t_j + 4(n+1-2r)\cot t_j$ and $\omega_{jk} = \cos 2t_j - \cos 2t_k$.

For the above solution A_j, B_{jk}, we have by the condition (B)

$$C_j = \frac{1}{\Omega}(A_j \Omega)_{t_j} + \frac{1}{2\Omega} \sum_{j<k} \{(B_{jk}\Omega)_{t_k} + (B_{jk}\Omega)_{t_j}\} - \frac{1}{2\Omega} \sum_{j\neq k} \Omega_{t_j t_k t_k}. \tag{4.8}$$

We define an operator Q_1 by the right hand side of the equation (4.2), where the coefficients A_j, B_{jk}, and C_j are given by the functions (4.6), (4.7), and (4.8), respectively. Then a differential operator $Q_2 := \mathrm{rad}(P) - Q_1$ is a second order differential operator and satisfies the conditions (B), (C), and (D).

Here we have the following lemma.

Lemma 4.3. *If a second order differential operator Q satisfies the condition (B), (C), and (D), then Q can be written as $c\,\mathrm{rad}(\Delta_{G/K_1})$ for a suitable constant c.*

Since the above lemma can be proved by the same argument as in Lemma 5.4 of Chapter 1 or in Lemma 5.2 of Chapter 2, we omit the proof.

By Lemma 4.3, the fourth order differential operator Q which satisfies the above four conditions (A), (B), (C), and (D) can be written as $Q_1 + c\,\mathrm{rad}(\Delta_{G/K_l})$ for some constant c. Therefore, by Lemma 4.2, we have the following proposition.

Proposition 4.4. *There is a constant c such that* $16\,\mathrm{rad}(L_c) = Q_1$

§5. Proof of the main theorem.

In this section, we prove the main theorem that we mentioned in the introduction of this chapter and we will show that the range of the Radon transform R is charcterized by the differential operator L_c for some constant c.

We denote by $a(m_1, \ldots, m_r)$ the eigenvalue of L_c on $V(m_1, \quad, m_r)$, where we choose a constant c such that L_c satisfies Proposition 4.4, that is, $16\,\mathrm{rad}(L_c) = Q_1$. Let $\phi_{(m_1,\ldots,m_r)}$ be the zonal spherical function which belongs to $V(m_1, \ldots, m_r)$. We denote by $u_{(m_1,\ldots,m_r)}$ the restriction of $\phi_{(m_1,\ldots,m_r)}$ to the Weyl chamber $\mathcal{A}^+$

Lemma 5.1. ([12], Theorem 8.1.) *The function $u_{(m_1,\ldots,m_r)}$ has a Fourier series expansion on $\mathcal{A}^+$ of the form*

$$
\begin{aligned}
&u_{(m_1,\ldots,m_r)}(t_1, \ldots, t_r) \\
&= \sum_{\substack{\lambda \leq m_1 M_1 + \cdots + m_r M_r \\ \lambda \in Z(G,K), \text{finite sum}}} \eta_\lambda \exp \sqrt{-1}(\lambda, t_1 H_1 + \cdots + t_r H_r),
\end{aligned}
$$

where $\eta_{m_1 M_1 + \cdots + m_r M_r} > 0$.

Let f_1 and f_2 be Fourier series on $\mathcal{A}^+$ of the form

$$
f_1 = \sum_{\lambda \leq \Lambda_1, \lambda \in Z(G,K)} \zeta_\lambda \exp \sqrt{-1}(\lambda, t_1 H_1 + \cdots + t_r H_r),
$$

$$
f_2 = \sum_{\lambda \leq \Lambda_2, \lambda \in Z(G,K)} \bar{\zeta}_\lambda \exp \sqrt{-1}(\lambda, t_1 H_1 + \cdots + t_r H_r).
$$

We denote $f_1 \sim f_2$ when $\Lambda_1 = \Lambda_2$ and $\zeta_{\Lambda_1} = \bar{\zeta}_{\Lambda_2} (\neq 0)$. Obviously, the relation $\sim$ is an equivalence relation.

Lemma 5.2. *We have the following relations.*

$$
\begin{aligned}
\sigma_{t_j} &\sim 2\sqrt{-1}(n+2-2r)\sigma, \\
\omega_{t_j} &\sim 2\sqrt{-1}(r-j)\omega, \\
\frac{\partial}{\partial t_j} u_{(m_1,\ldots,m_r)} &\sim 2\sqrt{-1} l_j u_{(m_1,\ldots,m_r)}, \\
\Omega_{t_j} &\sim \sqrt{-1}(2n+5-4j)\Omega, \\
A_j\Omega^2 &\sim -8(2n+3-4j)(j-1)\Omega^2, \\
B_{jk}\Omega^2 &\sim -(2n+5-4j)(2n+5-4k)\Omega^2, \\
C_j\Omega^3 &\sim -\sqrt{-1}(2n+3-4j)(j-1)(2n+5-4j)\Omega^3,
\end{aligned}
\tag{5.1}
$$

where the functions A_j, B_{jk}, and C_j are given by (4.5), (4.6), *and* (4.7), *respectively, and l_j is given as follows.*

$$
\begin{aligned}
(n+1 > 2r):&\quad l_j = m_j + \cdots + m_r \\
(n+1 = 2r):&\quad l_j = m_j + \cdots + m_{r-1} + 2m_r.
\end{aligned}
\tag{5.2}
$$

Theorem 5.3. *The eigenvalue $a(m_1,\ldots,m_r)$ of L_c on $V(m_1,\ldots,m_r)$ is given by the formula*

$$
\begin{aligned}
a(m_1,\ldots,m_r) = &\sum_{1\le j<k\le r} l_j l_k (l_j + 2n + 5 - 4j)(l_k + 2n + 5 - 4k) \\
&+ 2\sum_{j=2}^{r} (j-1)(2n+3-4j) l_j (l_j + 2n + 5 - 4j),
\end{aligned}
$$

where l_j is given by (5.2).

Proof. By definition, we have

$$
16 L_c \phi_{(m_1,\ldots,m_r)} = 16 a(m_1,\ldots,m_r)\phi_{(m_1,\ldots,m_r)}. \tag{5.3}
$$

We restrict both sides of (5.3) to the Weyl chamber $\mathcal{A}^+$, and then we have

$$
\mathrm{rad}(L_c) u_{(m_1,\ldots,m_r)} = a(m_1,\ldots,m_r) u_{(m_1,\ldots,m_r)}.
$$

On the other hand, by Proposition 4.5 and Lemma 5.2, we have

$$
\begin{aligned}
&16\Omega^3 \operatorname{rad}(L_c)u_{(m_1,\dots,m_r)}\\
&= \Omega^3 Q_1 u_{(m_1,\dots,m_r)}\\
&= \Omega^3 \sum_{j<k} \frac{\partial^4}{\partial t_j{}^2 \partial t_k{}^2} u_{(m_1,\dots,m_r)} + \Omega^3 \sum_{j\neq k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial t_j{}^2 \partial t_k} u_{(m_1,\dots,m_r)}\\
&+ \Omega^3 \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial t_j{}^2} u_{(m_1,\dots,m_r)} + \Omega^3 \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} u_{(m_1,\dots,m_r)}\\
&+ \Omega^3 \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j} u_{(m_1,\dots,m_r)}\\
&\sim \{ \sum_{1\leq j<k\leq r} 16 l_j l_k (l_j + 2n + 5 - 4j)(l_k + 2n + 5 - 4k)\\
&\quad + \sum_{j=2}^{r} 32(j-1)(2n+3-2j) l_j (l_j + 2n + 5 - 4j) \} \, \Omega^3 u_{(m_1,\dots,m_r)}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
a(m_1,\dots,m_r) = &\sum_{1\leq j<k\leq r} l_j l_k (l_j + 2n + 5 - 4j)(l_k + 2n + 5 - 4k)\\
&+ \sum_{j=2}^{r} (j-1)(2n+3-2j) l_j (l_j + 2n + 5 - 4j)
\end{aligned}
$$

∎

The following corollary is obvious.

Corollary 5.4. *$V(m_1,\dots,m_r)$ is contained in* $\operatorname{Ker} P$ *if and only if $m_2 = \cdots = m_r = 0$.*

By the same calculas as in the proof of Theorem 5.3, we get the following proposition.

Proposition 5.5. *The eigenvalue $\tau(m_1,\dots,m_r)$ of the Laplacian Δ_{G/K_l} on $V(m_1,\dots,m_r)$ is given by*

$$
\tau(m_1,\dots,m_r) = \frac{1}{2(n+2)} \sum_{j=1}^{r} l_j (l_j + 2n + 5 - 4j) \tag{5.11}
$$

We prove the main theorem of this chapter by showing the following theorem.

Theorem 5.6. *There exists a constant c such that the range* $\operatorname{Im} R$ *of the Radon transform R is identical with the kernel of the invariant differential operator L_c on $Gr(l,n;\mathbb{H})$ defined by* (4.1), *that is,* $\operatorname{Ker} L_c = \operatorname{Im} R$.

Proof. Let $V := \bigoplus_{m=0}^{\infty} V(m, 0, \ldots, 0)$ and $\tilde{V} := \bigoplus_{m=0}^{\infty} V_m$ (direct sum). Then we have $R : \tilde{V} \to V$ and $S : V \to \tilde{V}$ Here, S is the inversion map defined in (3.1). Moreover, we have $SR = Id$ on $\tilde{V}$ and $RS = Id$ on V by Proposition 3.2 and Corollary 3.3. On the other hand, by Collorary 5.4, V is dense in $\operatorname{Ker} P$ in C^{∞}-topology.

Since the inversion map $S : C^{\infty}(M) \to C^{\infty}(\mathbf{P}^n\mathbf{H})$ is continuous, we have $RS = Id$ on $\operatorname{Ker} P$ This completes the proof. ∎

Remark 5.7. We do not obtain the explicit form of the range characterizing operator in the same manner as in Chapter 1 or in Chapter 2. Moreover, we do not give the constant c of L_c in Theorem 5.6 explicitly. Indeed, it is too cumbersome and seems meaningless for the author to compute the above constant c.

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